Controle Digital
Parte II

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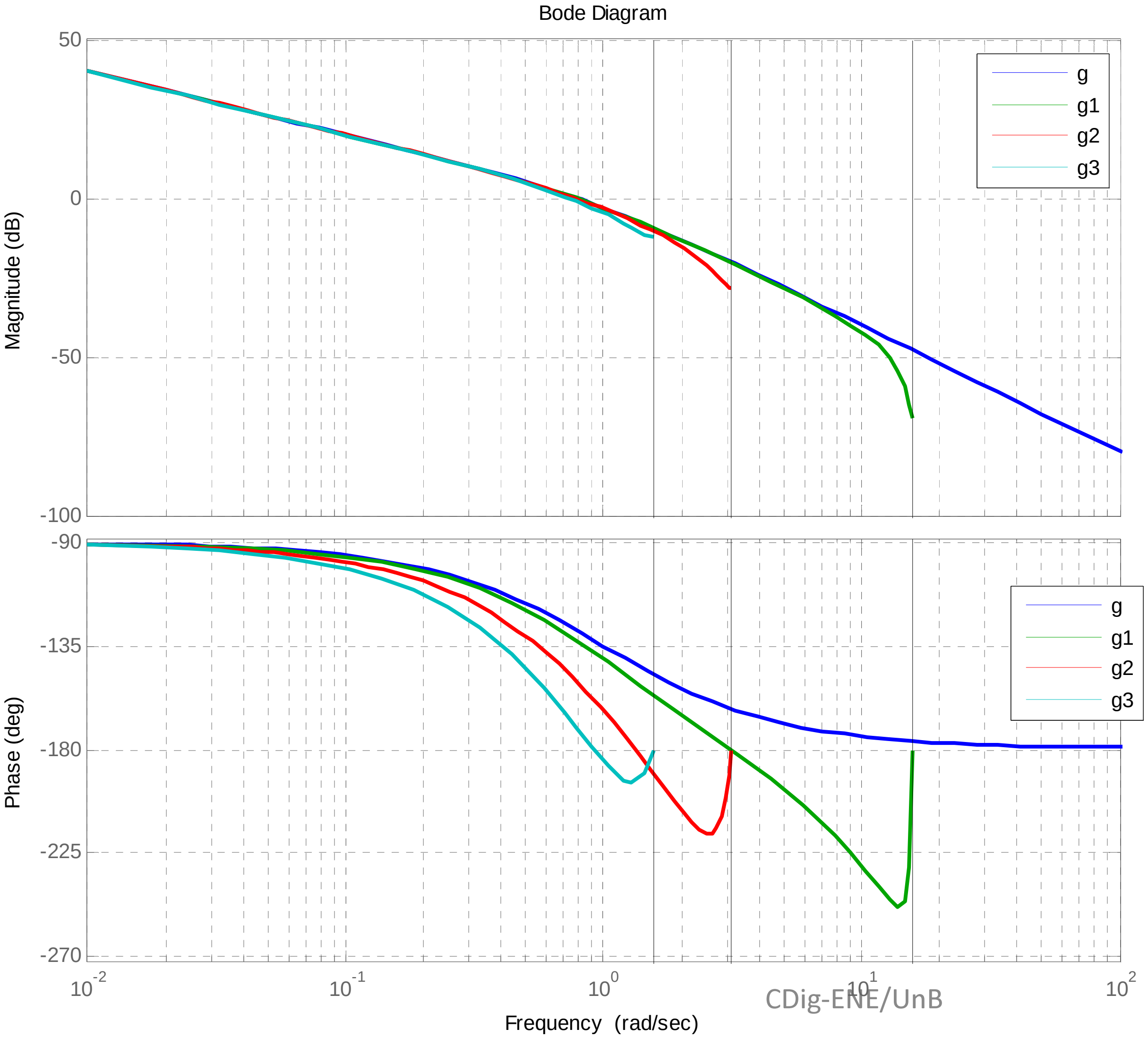
ENE/FT/UnB

7.4 Métodos no Domíno da Frequência



$$g(s) = \frac{1}{s^2 + s}$$

$$g1(s) = c2d(g, 0.2)$$

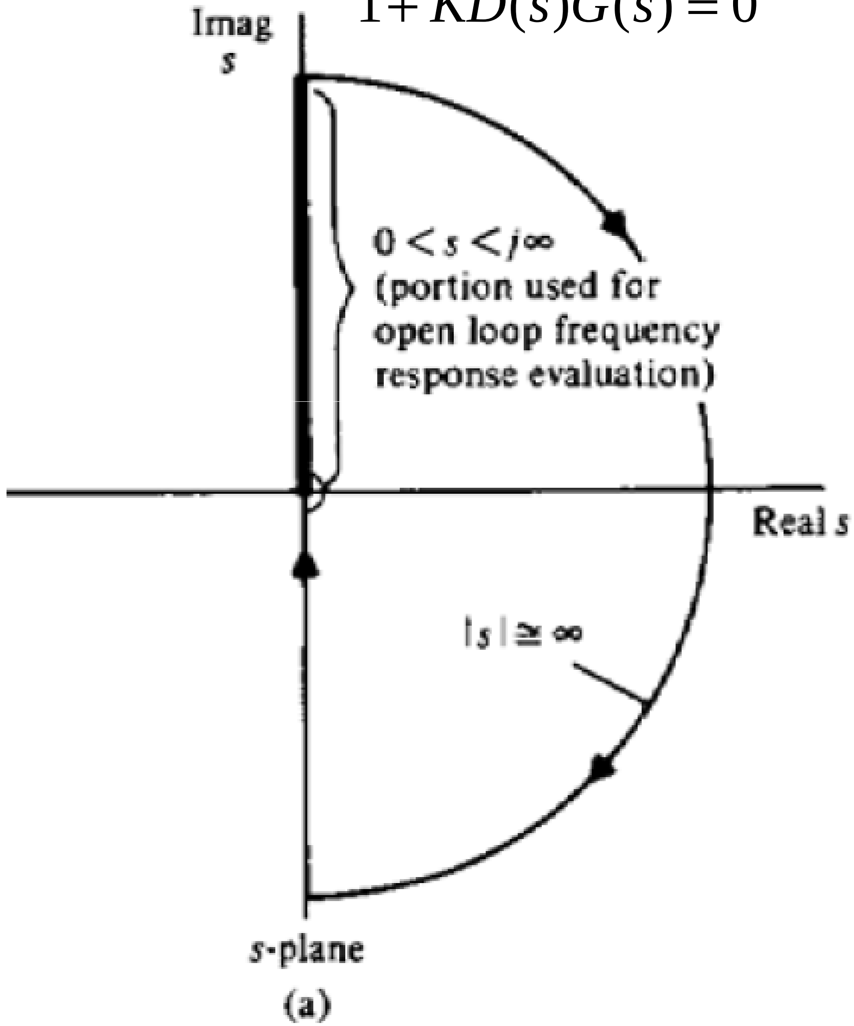
$$g2(s) = c2d(g, 1.0)$$

$$g2(s) = c2d(g, 2.0)$$

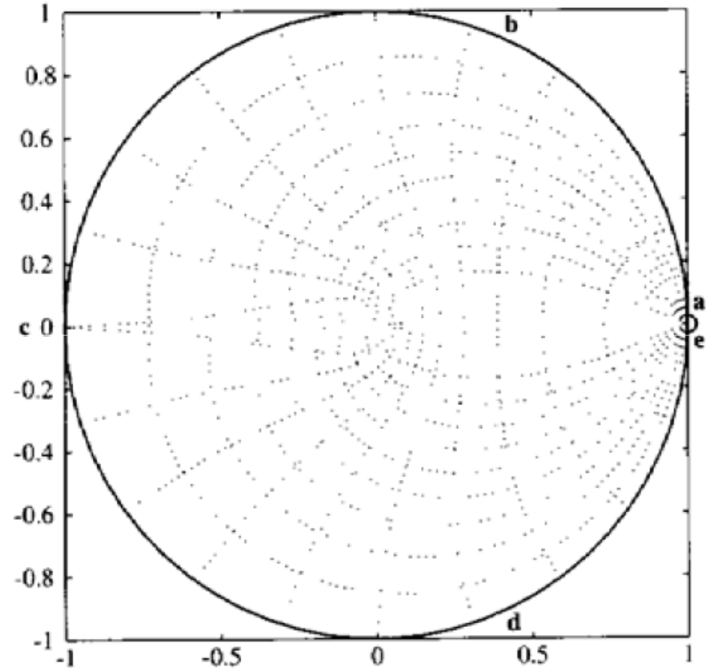
$$\Delta\phi = \frac{\omega T}{2}$$

Critério de Estabilidade de Nyquist

$$1 + KD(s)G(s) = 0$$



$$1 + KD(z)G(z) = 0$$



$$Z = P + N$$

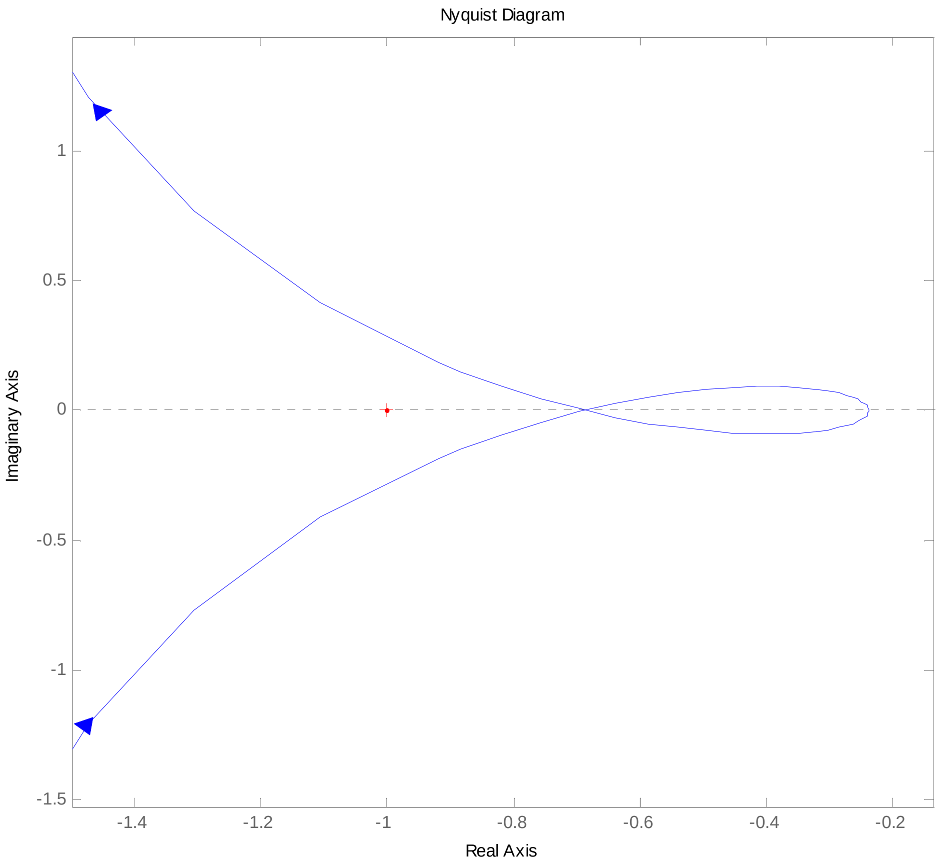
$Z -$ *raízes da eq. carac fora do circ. unit.*

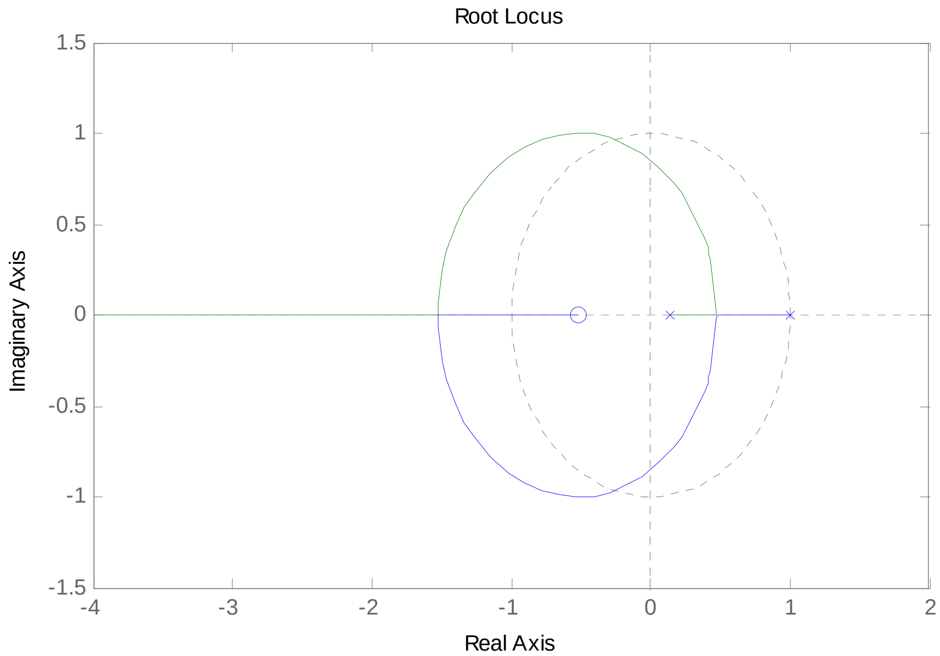
$P -$ *# pólos fora circ. unit. em MA*

$N -$ *# envolvimentos* (-1) *sentido horário*

Exemplo - Estabilidade via Nyquist

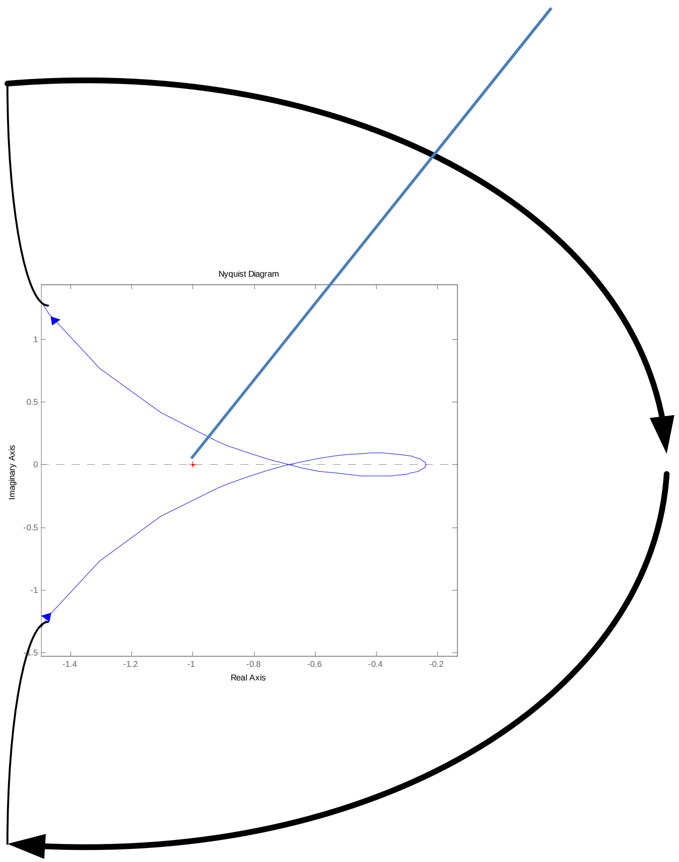
$$G(z) = \frac{1.1353\,(z + 0.5232)}{(z - 1)\,(z - 0.1353)}$$





$$1 + KD(z)G(z) = 0 \qquad Z = P + N$$

Exemplo - Estabilidade via Nyquist



$$G(z) = \frac{1.1353\,(z + 0.5232)}{(z - 1)\,(z - 0.1353)}$$

$$1 + KD(z)G(z) = 0$$

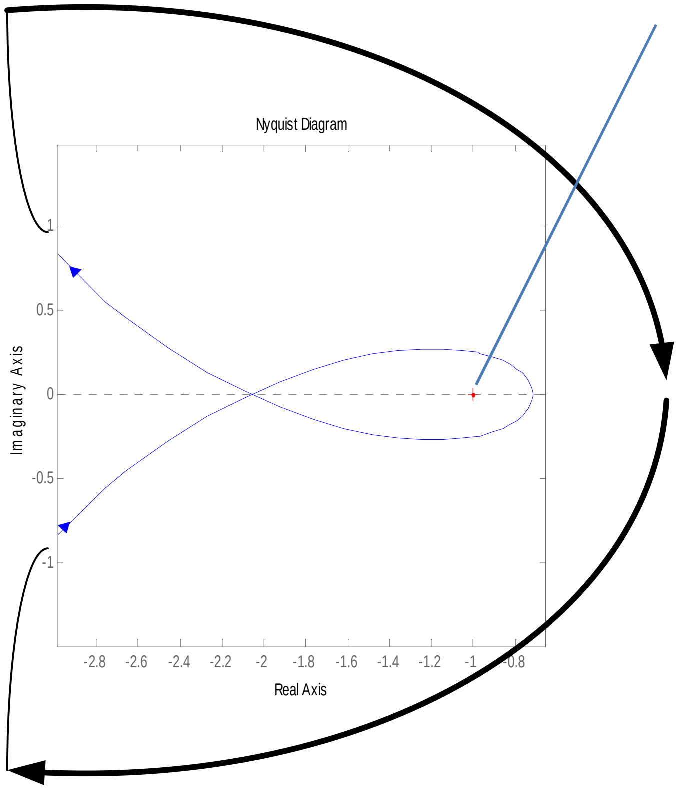
$$Z = P + N$$

$$N = 0$$

$$P = 0$$

$$Z = 0 \Rightarrow Sistema\ \ Estável$$

Exemplo - Estabilidade via Nyquist



$$G(z) = 3\frac{1.1353\,(z + 0.5232)}{(z - 1)\,(z - 0.1353)}$$

$$1 + KD(z)G(z) = 0$$

$$Z = P + N$$

$$N = 2$$

$$P = 0$$

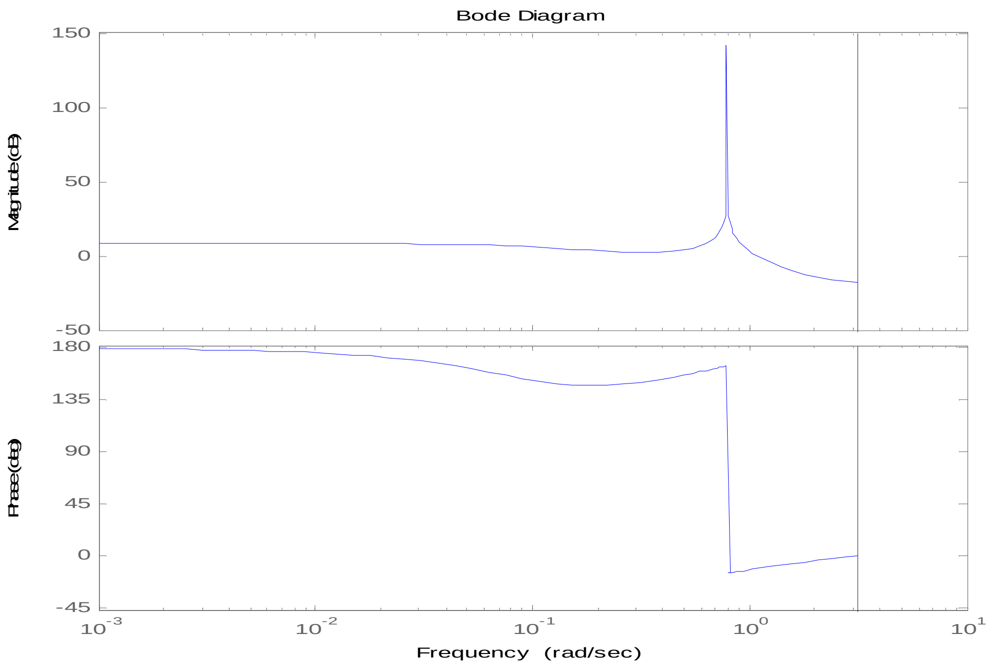
$$Z = 2$$

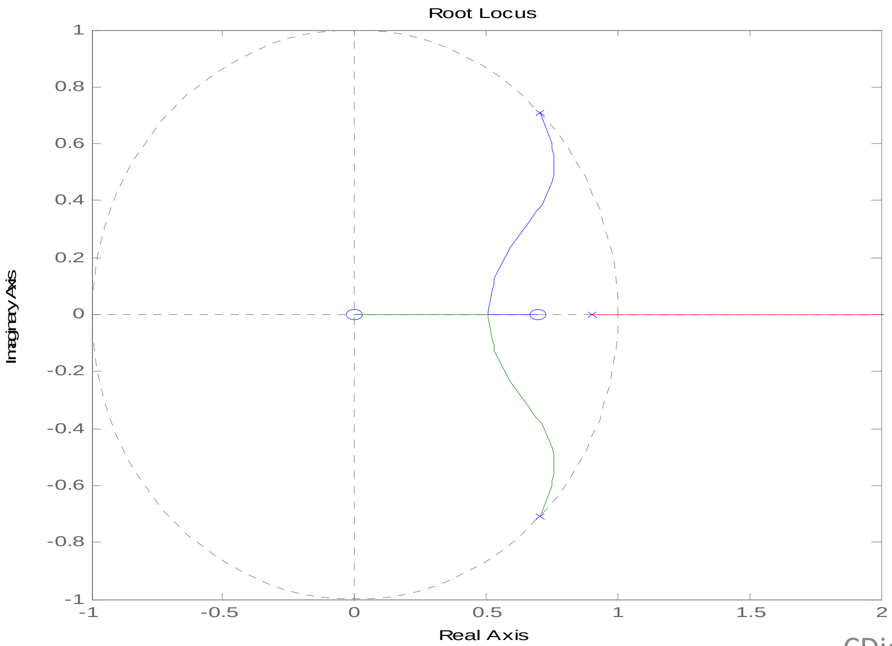
Dois zeros da eq. característica fora do circulo unitário = dois pólos de MF fora do círculo unitário

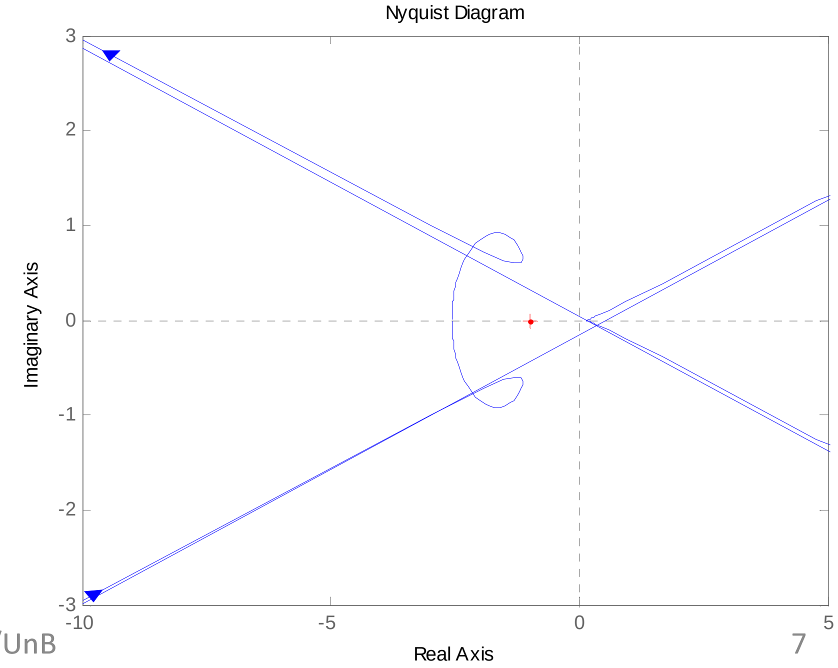
$\Rightarrow$ *Sistema Instável*

Exemplo

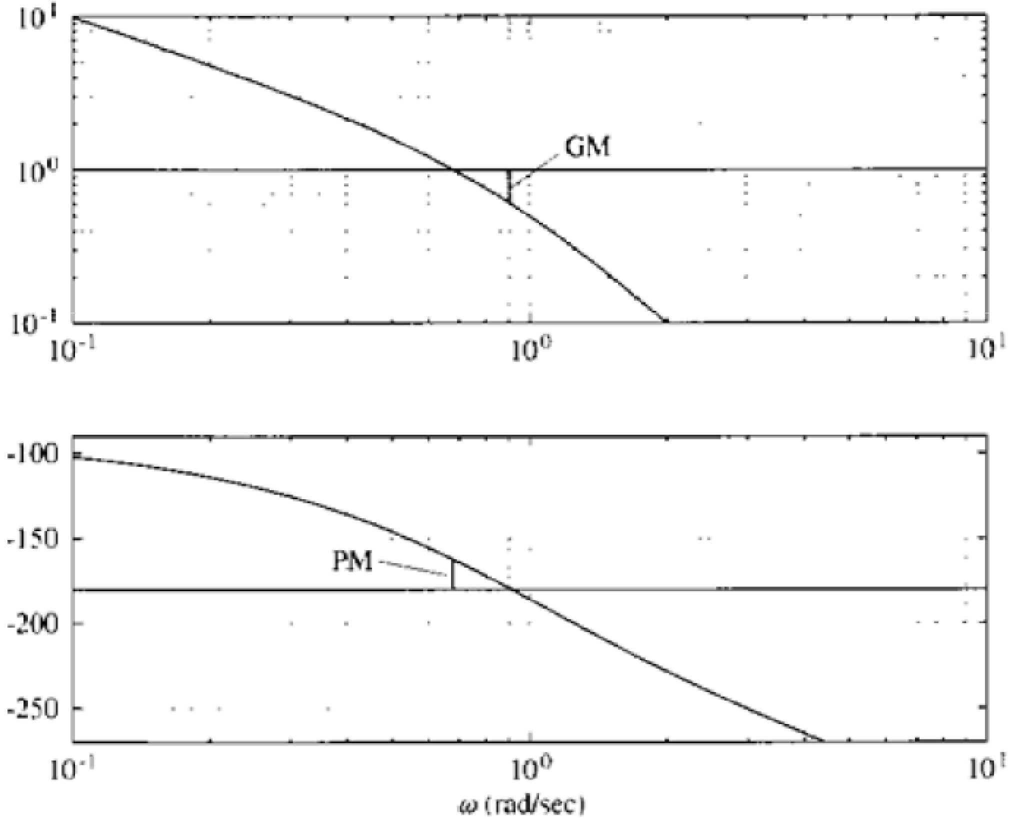
$$G(z) = \frac{-0.5\,z\,(z-0.7)}{(z-0.9)\,(z\hat{}2-1.414z+1)}$$

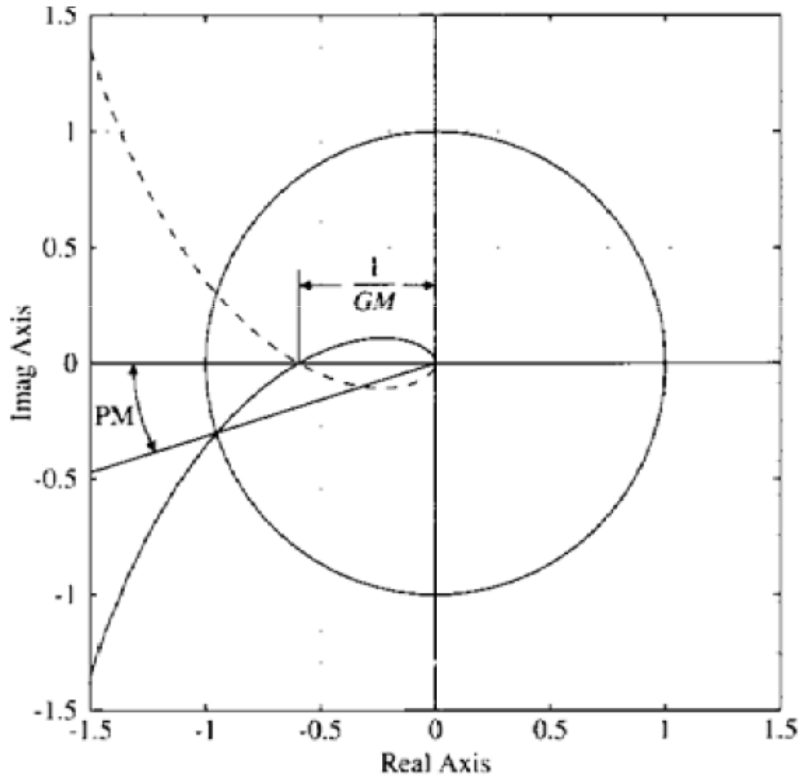






Margem de Ganho e Margem de Fase



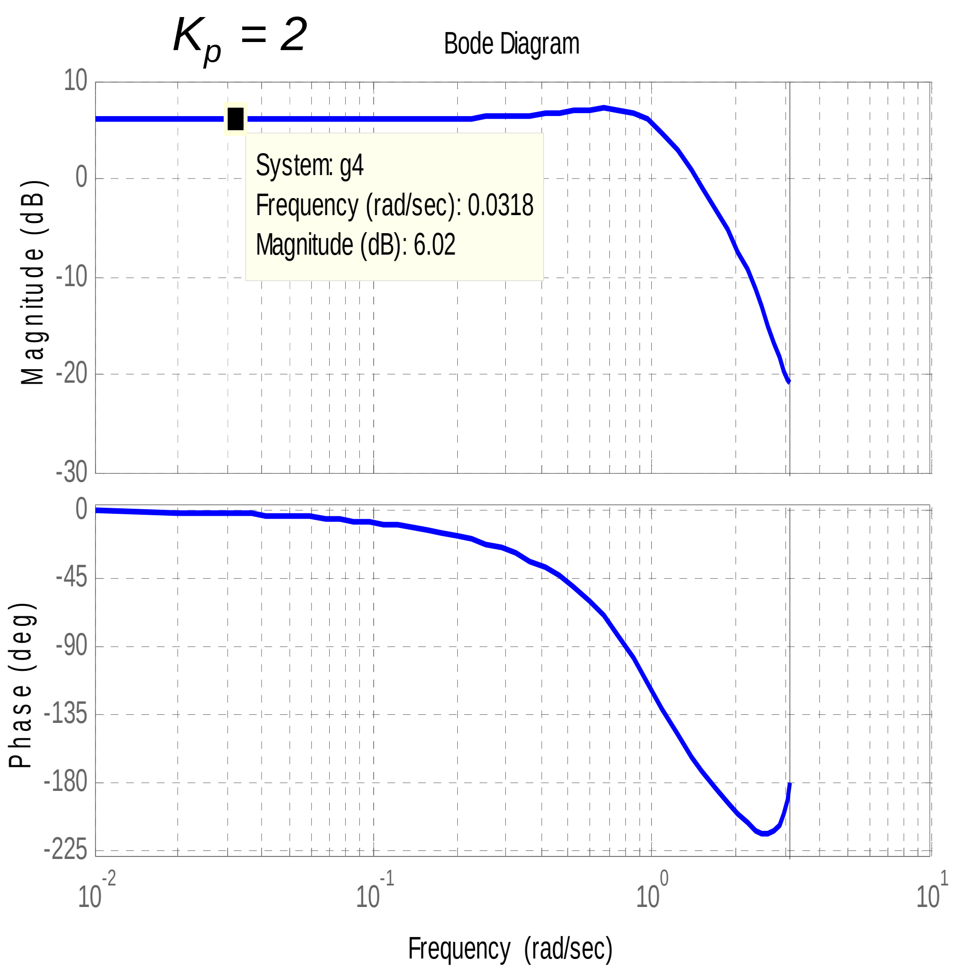


$$\zeta \cong \frac{PM}{100}$$

Coeficientes de Erro (baixa freq.)

Sistema tipo 0

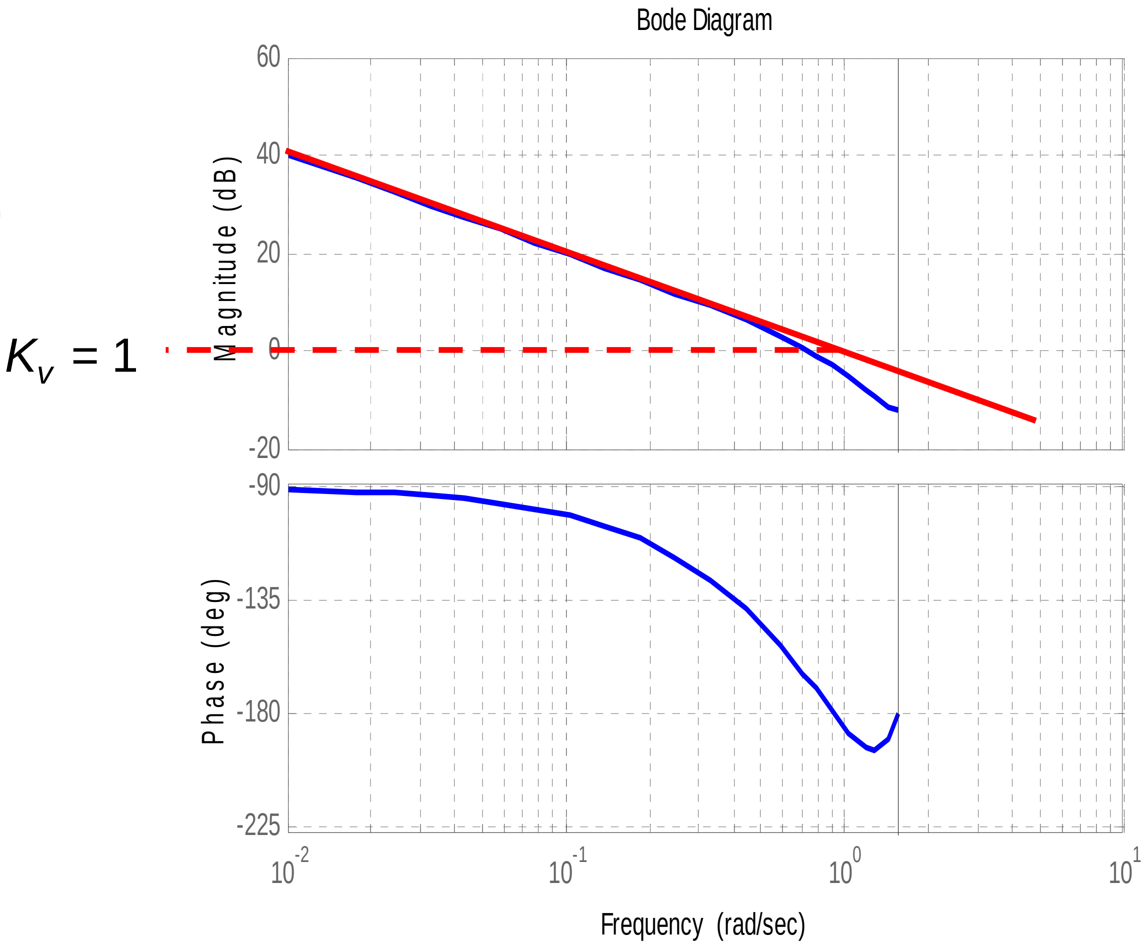
$$K_p = \lim_{z \to 1} D(z)G(z)$$



Coeficientes de Erro (baixa freq.)

Sistema tipo 1

$$K_v = \lim_{z \to 1} \frac{(z-1)D(z)G(z)}{Tz}$$



Controle no Espaço de Estados

- Emulação (em ”s”)
- ➢ Projeto direto
  - ➢ Obtenção do Modelo Discreto Equivalente

$$Eq.\ de\ Estados \begin{cases} \dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u \\ y = \mathbf{H}\mathbf{x} + \mathbf{J}u \end{cases}$$

Solução Homogênea $\dot{\mathbf{x}}_h = \mathbf{F}\mathbf{x}_h(t), \qquad \mathbf{x}_h(t_0) = \mathbf{x}_0$

$$\mathbf{x}_h(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0) \qquad e^{\mathbf{F}(t-t_0)} = \mathbf{I} + \mathbf{F}(t-t_0) + \mathbf{F}^2\frac{(t-t_0)^2}{2!} + \mathbf{F}^3\frac{(t-t_0)^3}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} \mathbf{F}^k \frac{(t-t_0)^k}{k!}.$$

(*Exponencial da Matriz*)

Propriedades – Exponencial da Matriz

$$\mathbf{x}_h(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0)$$

$$\begin{cases} \mathbf{x}(t_1) = e^{\mathbf{F}(t_1-t_0)}\mathbf{x}(t_0) \\ \mathbf{x}(t_2) = e^{\mathbf{F}(t_2-t_0)}\mathbf{x}(t_0) \end{cases} \Rightarrow \quad \mathbf{x}(t_2) = e^{\mathbf{F}(t_2-t_1)}x(t_1)$$

$$\mathbf{x}(t_2) = e^{\mathbf{F}(t_2-t_1)}e^{\mathbf{F}(t_1-t_0)}\mathbf{x}(t_0)$$

$$e^{\mathbf{F}(t_2-t_0)} = e^{\mathbf{F}(t_2-t_1)}e^{\mathbf{F}(t_1-t_0)}$$

$$\mathbf{I} = e^{-\mathbf{F}(t_1-t_0)}e^{\mathbf{F}(t_1-t_0)}$$

Solução Completa

$$\mathbf{x}(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau)d\tau$$

Para um período de amostragem:

$t = kT + T$ and $t_0 = kT$

$$\mathbf{x}(kT + T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

ZOH: $u(\tau) = u(kT), \qquad kT \le \tau < kT + T$

Solução Completa

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

ZOH: $u(\tau) = u(kT), \qquad kT \le \tau < kT + T$

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_{0}^{T} e^{\mathbf{F}\eta}d\eta\mathbf{G}u(kT) \qquad \eta = kT + T - \tau$$

Definindo:

$$\mathbf{\Phi} = e^{\mathbf{F}T}$$

$$\mathbf{\Gamma} = \int_{0}^{T} e^{\mathbf{F}\eta}d\eta\mathbf{G}.$$

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k) \\ y(k) &= \mathbf{H}\mathbf{x}(k) + Ju(k) \end{aligned}$$

Espaço de Estados Discreto Equivalente ZOH

Exemplo EE

$$G(s) = \frac{1}{s^2} \qquad \ddot{\theta} = u.$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{G}} u.$$

$$\theta = y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k) \\ y(k) &= \mathbf{H}\mathbf{x}(k) + Ju(k) \end{aligned}$$

$$\begin{aligned} \mathbf{\Phi} &= \mathbf{I} + \mathbf{F}T + \frac{\mathbf{F}^2T^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Cálculo das Matrizes

$$\mathbf{\Phi} = \mathbf{I} + \mathbf{F}T\mathbf{\Psi}$$

onde $$\mathbf{\Psi} = \mathbf{I} + \frac{\mathbf{F}T}{2!} + \frac{\mathbf{F}^2T^2}{3!} + \cdots$$

$$\begin{aligned}\mathbf{\Gamma} &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^{k+1}}{(k+1)!}\mathbf{G} \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^k}{(k+1)!} T\mathbf{G} \\ &= \mathbf{\Psi} T \mathbf{G}.\end{aligned}$$

Cálculo das Matrizes

$$\Gamma = \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^{k+1}}{(k+1)!}\mathbf{G} = \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^k}{(k+1)!} T\mathbf{G} = \Psi T \mathbf{G}.$$

onde

$$\Psi = \mathbf{I} + \frac{\mathbf{F}T}{2!} + \frac{\mathbf{F}^2 T^2}{3!} + \cdots$$

$$\Phi = \mathbf{I} + \mathbf{F}T + \frac{\mathbf{F}^2 T^2}{2!} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

$$\Gamma = \left[\mathbf{I}T + \mathbf{F}\frac{T^2}{2!} + \frac{\mathbf{F}^2 T^3}{3!}\right]\mathbf{G} = \left\{\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\frac{T^2}{2}\right\}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

Exemplo: Função de Transferência

$$\frac{Y(z)}{U(z)} = \mathbf{H}[z\mathbf{I} - \mathbf{\Phi}]^{-1}\mathbf{\Gamma}$$

$$\frac{Y(z)}{U(z)} = [1 \quad 0]\left\{z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\right\}^{-1}\begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$
$$= \frac{T^2}{2}\frac{(z+1)}{(z-1)^2}.$$

Controle no Espaço de Estados

- Emulação (em "s")
  - Projeto direto
    - Modelo Discreto Equivalente

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u$$
$$y = \mathbf{H}\mathbf{x} + \mathbf{J}u$$

ZOH

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k) \\ y(k) &= \mathbf{H}\mathbf{x}(k) + Ju(k) \end{aligned}$$

Onde:

$$\begin{aligned} \mathbf{\Phi} &= e^{\mathbf{F}T}, \\ \mathbf{\Gamma} &= \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G} \end{aligned}$$

Projeto da Lei de Controle

- Realimentação de Estados

$$u = -\mathbf{K}\mathbf{x} = -\begin{bmatrix} K_1 & K_2 & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) - \mathbf{\Gamma}\mathbf{K}\mathbf{x}(k)$$

$$(z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K})\mathbf{X}(z) = 0$$

$$|zI - \Phi + \Gamma K| = 0$$ Eq. característica

$$\alpha_c(z) = (z - \beta_1)(z - \beta_2)\cdots(z - \beta_n) = 0$$ Eq. Característica desejada → igualar potências

Exemplo

$$G(s) = \frac{1}{s^2} \qquad \ddot{\theta} = u.$$

$$\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

Equação característica desejada: $z^2 - 1.6z + 0.70 = 0.$

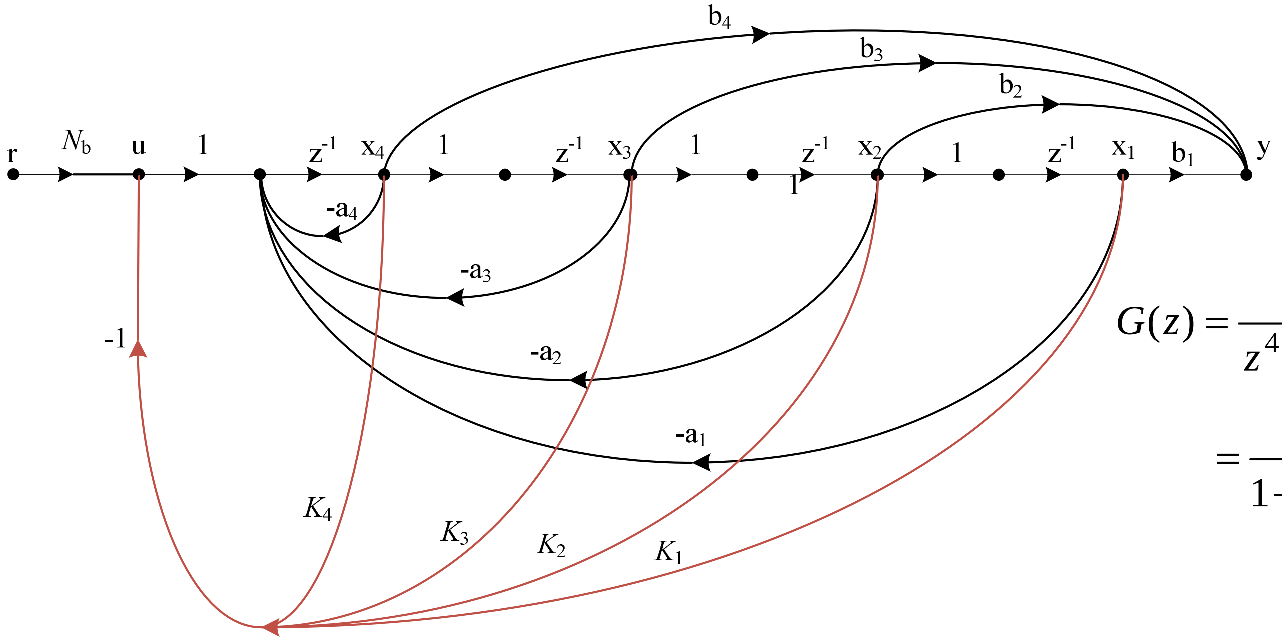
$$\left| z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} [K_1 \quad K_2] \right| = 0$$

$$z^2 + (TK_2 + (T^2/2)K_1 - 2)z + (T^2/2)K_1 - TK_2 + 1 = 0$$

$$TK_2 + (T^2/2)K_1 - 2 = -1.6.$$
$$(T^2/2)K_1 - TK_2 + 1 = 0.70.$$

$$K_1 = \frac{0.10}{T^2} = 10. \qquad K_2 = \frac{0.35}{T} = 3.5$$

Utilizando a forma canônica controlável



$$G(z) = \frac{b_4 z^3 + b_3 z^2 + b_2 z + b_1}{z^4 + a_4 z^3 + a_3 z^2 + a_2 z + a_1}$$

$$= \frac{b_4 z^{-1} + b_3 z^{-2} + b_2 z^{-3} + b_1 z^{-4}}{1 + a_4 z^{-1} + a_3 z^{-2} + a_2 z^{-3} + a_1 z^{-4}}$$

$$Eq.\ desejada\ \alpha(z) = z^4 + \alpha_4 z^3 + \alpha_3 z^2 + \alpha_2 z + \alpha_1$$

$$\Phi_c - \Gamma_c K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_1 - K_1 & -a_2 - K_2 & -a_3 - K_3 & -a_4 - K_4 \end{bmatrix}$$

$$\det(zI - \Phi_c + \Gamma_c K) = z^4 + (a_4 + K_4) z^3 + (a_3 + K_3) z^2 + (a_2 + K_2) z + (a_1 + K_1)$$

$$K_1 = \alpha_1 - a_1$$
$$K_2 = \alpha_2 - a_2$$
$$K_3 = \alpha_3 - a_3$$
$$K_4 = \alpha_4 - a_4$$

Controlabilidade

- Em malha aberta

$$|zI - \Phi| = 0$$

Forma canônica de Jordan (pólos distintos)

$$\mathbf{x}(k+1) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_n \end{bmatrix} u(k)$$

Controlabilidade → Nenhum Γ_i pode ser nulo

Posicionamento de pólos CACSD

- Para *s* e para *z*

  >> place

  >> acker

Fórmula de Ackermann (1972), SISO, até n=10, pólos repetidos

$$\mathbf{K} = \begin{bmatrix} 0 & \cdots & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Phi\Gamma} & \boldsymbol{\Phi}^2\boldsymbol{\Gamma} & \cdots & \boldsymbol{\Phi}^{n-1}\boldsymbol{\Gamma} \end{bmatrix}^{-1}}_{\text{Matriz de controlabilidade}} \alpha_c(\boldsymbol{\Phi})$$

Matriz de controlabilidade

$$\alpha_c(\boldsymbol{\Phi}) = \boldsymbol{\Phi}^n + \alpha_1\boldsymbol{\Phi}^{n-1} + \alpha_2\boldsymbol{\Phi}^{n-2} + \cdots + \alpha_n\mathbf{I}$$

$$\alpha_c(z) = |z\mathbf{I} - \boldsymbol{\Phi} + \boldsymbol{\Gamma}\mathbf{K}| = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n$$

Eq. Carcterística desejada

Posicionamento de pólos CACSD

- Place (Kautsky, Nichols & Von Dooren, 1985)

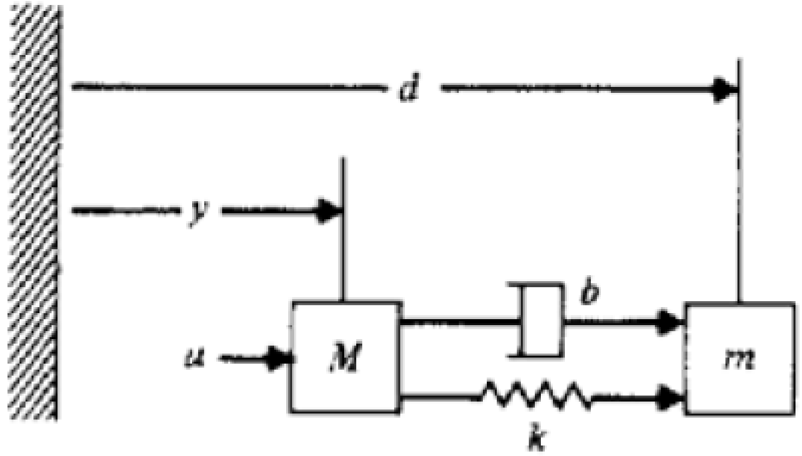
MIMO, ordem elevada, não aceita pólos repetidos

Exemplo: antena $1/s^2$

```
T = .1

Phi = [1 T;0 1]

Gam = [T^2/2;T]

p = [.8+i*.25;.8-i*.25]

K = acker(Phi,Gam,p)
```

$\mathbf{K} = [10.25 \quad 3.4875]$

Ex. Processo de 4ª ordem

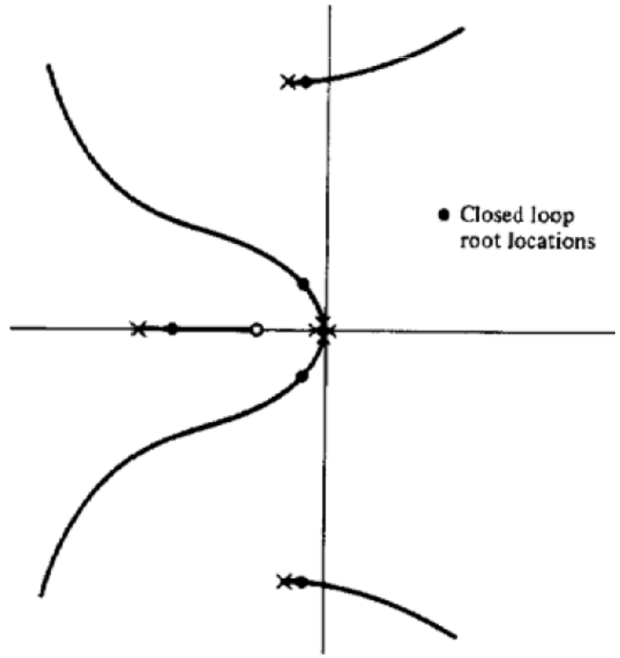


m, M = masses
b = damping coefficient
k = spring constant
F = applied control force

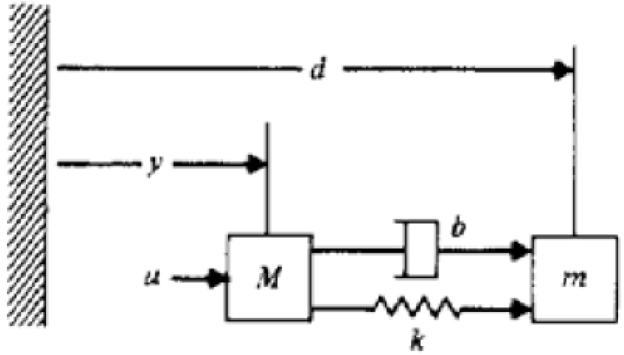
$$M\ddot{y} + (\dot{y} - \dot{d})b + (y - d)k = u.$$
$$m\ddot{d} + (\dot{d} - \dot{y})b + (d - y)k = 0.$$

$$\begin{bmatrix} Ms^2 + bs + k & -(bs + k) \\ -(bs + k) & ms^2 + bs + k \end{bmatrix} \begin{bmatrix} y(s) \\ d(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(s)$$

$$\frac{d(s)}{u(s)} = \frac{1}{M} \frac{\left(\frac{b}{m}s + \frac{k}{m}\right)}{s^2\left[s^2 + \left(1 + \frac{m}{M}\right)\left(\frac{b}{m}s + \frac{k}{m}\right)\right]}$$



Ex. 4ª ordem



$$\frac{y(s)}{u(s)} = \frac{1}{M}\,\frac{\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)}{s^2\left[s^2 + \left(1 + \frac{m}{M}\right)\left(\frac{b}{m}s + \frac{k}{m}\right)\right]} = G_5(s)$$

$$\frac{d(s)}{u(s)} \cong \frac{1}{M}\,\frac{k/m}{s^2\left[s^2 + \left(1 + \frac{m}{M}\right)\left(\frac{b}{m}s + \frac{k}{m}\right)\right]} = G_4(s)$$

Cont.

$\omega_n = 1$ rad/sec and damping $\zeta = 0.02$ and select a 10:1 ratio of the two masses. The parameters that provide these characteristics are: $M = 1$ kg, $m = 0.1$ kg, $b = 0.0036$ N-sec/m, and $k = 0.091$ N/m. Pick the sample rate to be 15 times faster than the resonance and show the free response to an initial condition of $d = 1$ m for two cases:

$$x = \begin{bmatrix} d & \dot{d} & y & \dot{y} \end{bmatrix}^T \qquad \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.91 & -0.036 & 0.91 & 0.036 \\ 0 & 0 & 0 & 1 \\ 0.091 & 0.0036 & -0.091 & -0.0036 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad J = 0.$$

The sample rate should be 15 rad/sec which translates to approximately $T = 0.4$ secs

Cont.

a)

```
sysD = c2d(sysC,T,'zoh')

p = [.9;.9;.9;.9]

[phi,gam,H,J] = ssdata(sysD)

K=acker(phi,gam,p)
```

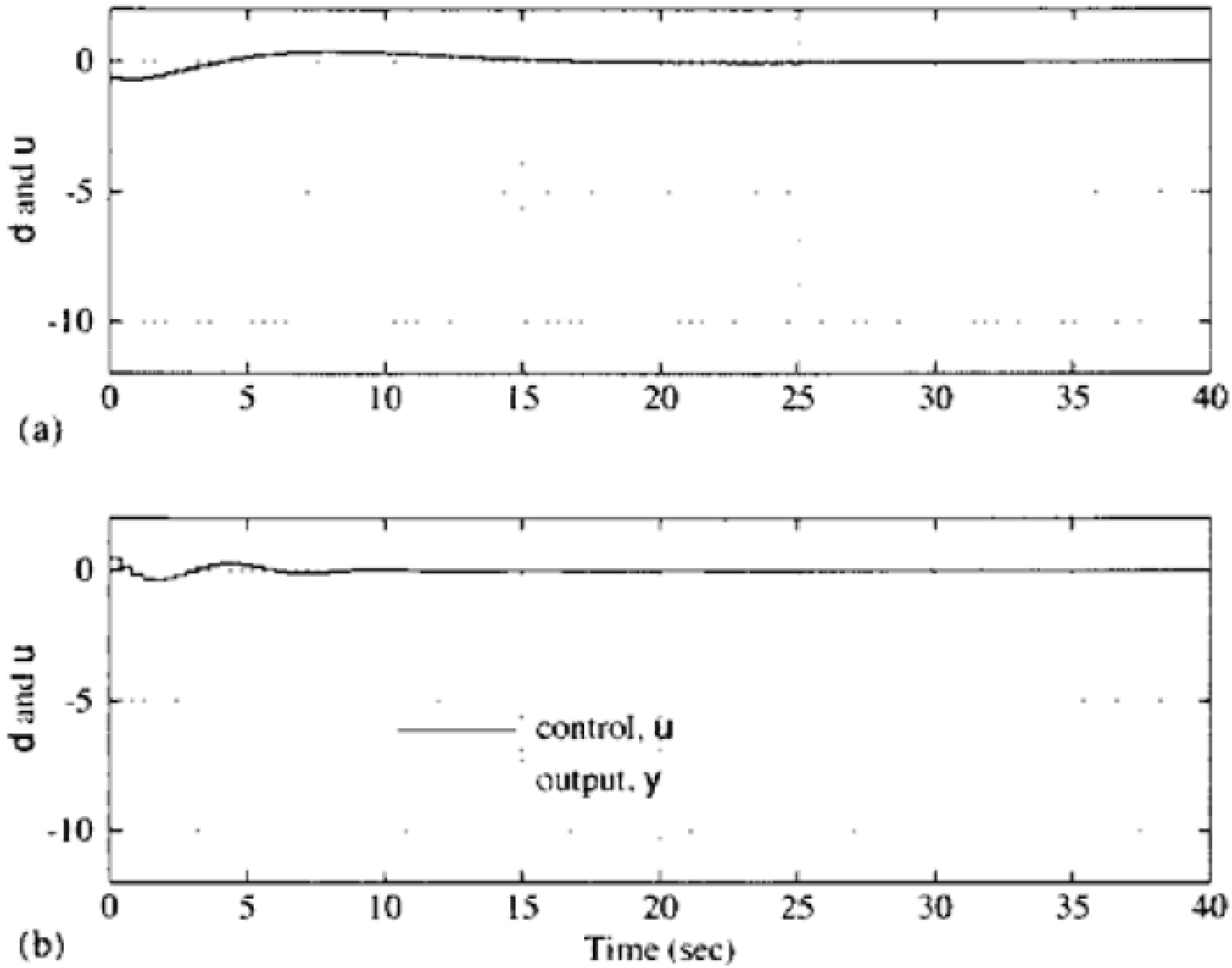
$\mathbf{K} = [0.650 \quad -0.651 \quad -0.645 \quad 0.718]$

```
sysCL = feedback(K*sysD,1)

Xo = [1;0;0;0]

y = initial(sysCL,Xo)
```

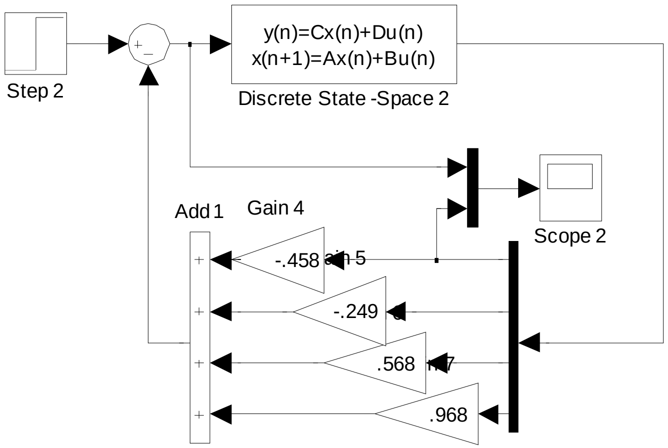
(b) For the desired poles at $z = 0.9 \pm j0.05, 0.8 \pm j0.4$

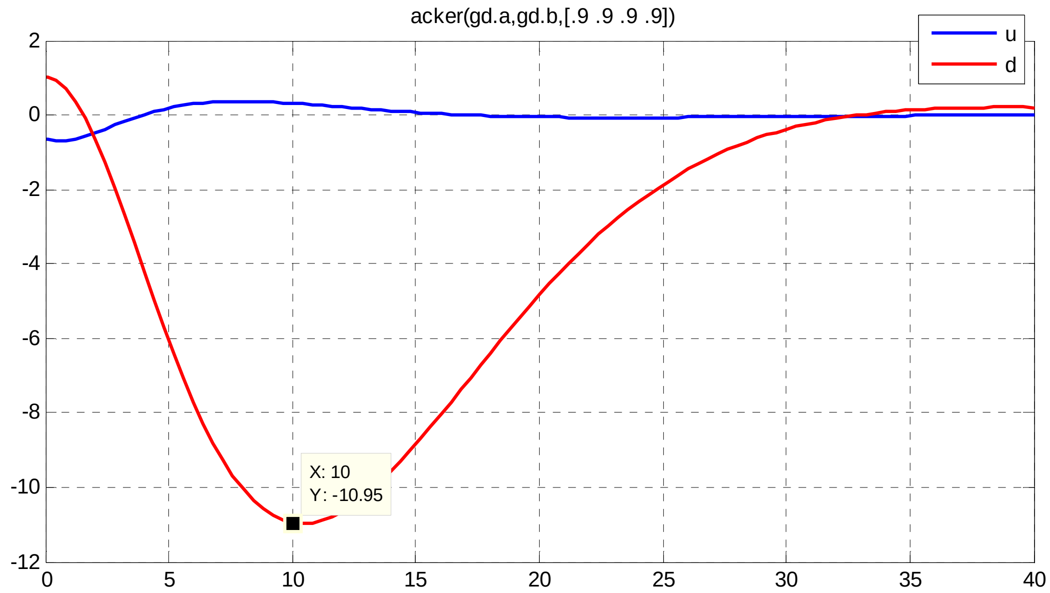
```
p = [.9+i*.05;.9-i*.05;.8+i*.4;.8-i*.4]
```

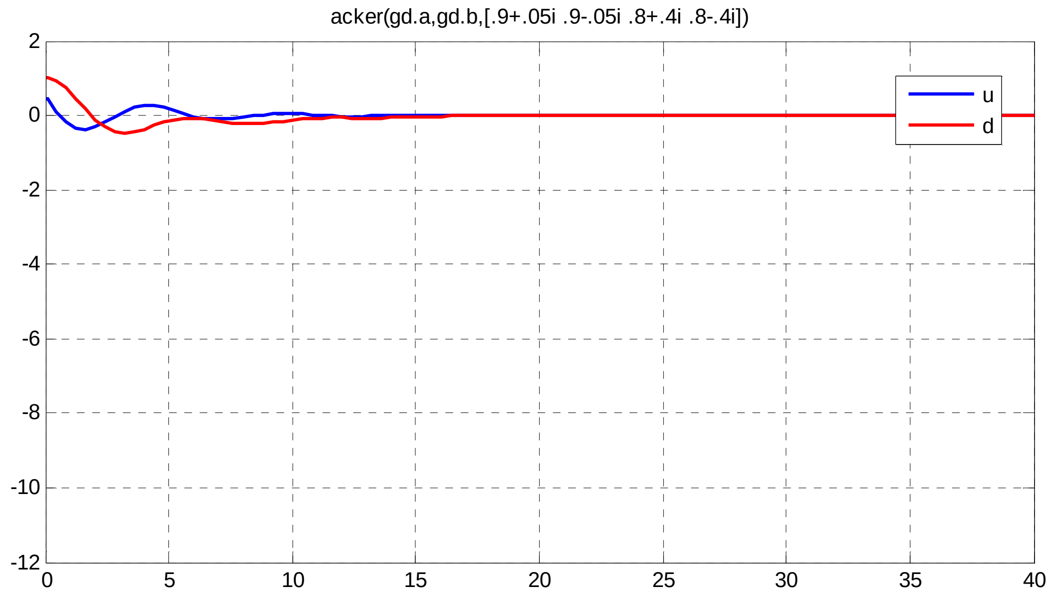
Resposta à condição inicial



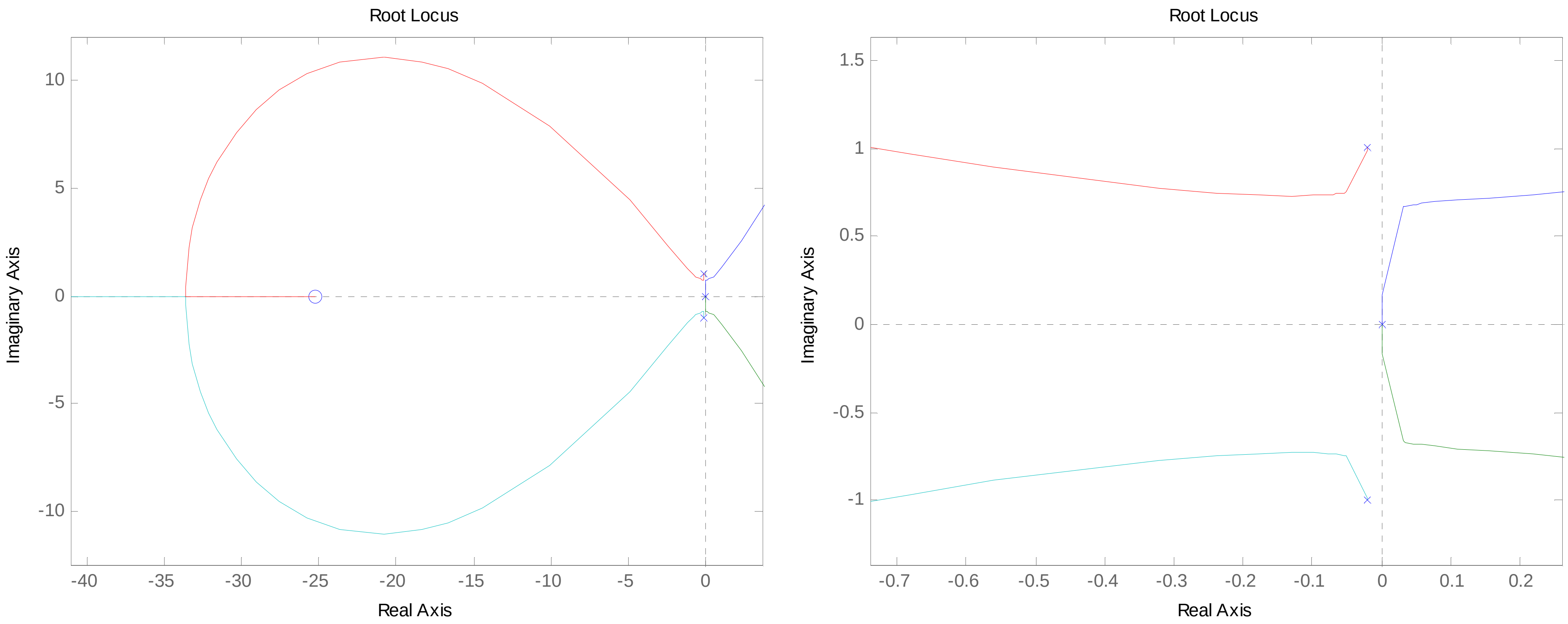
Simulação EE



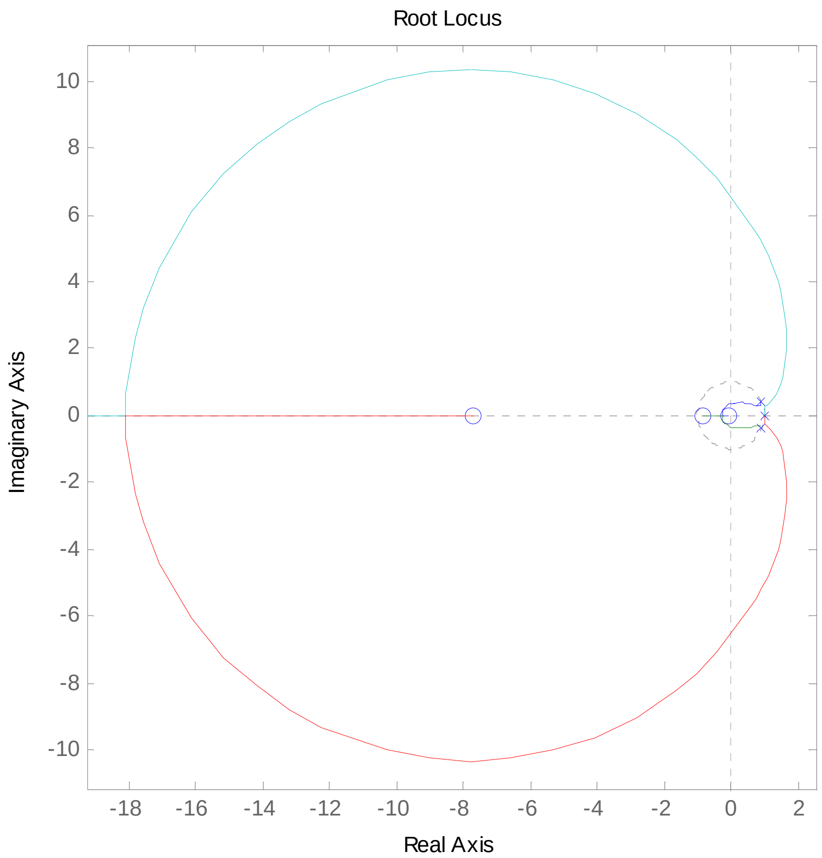


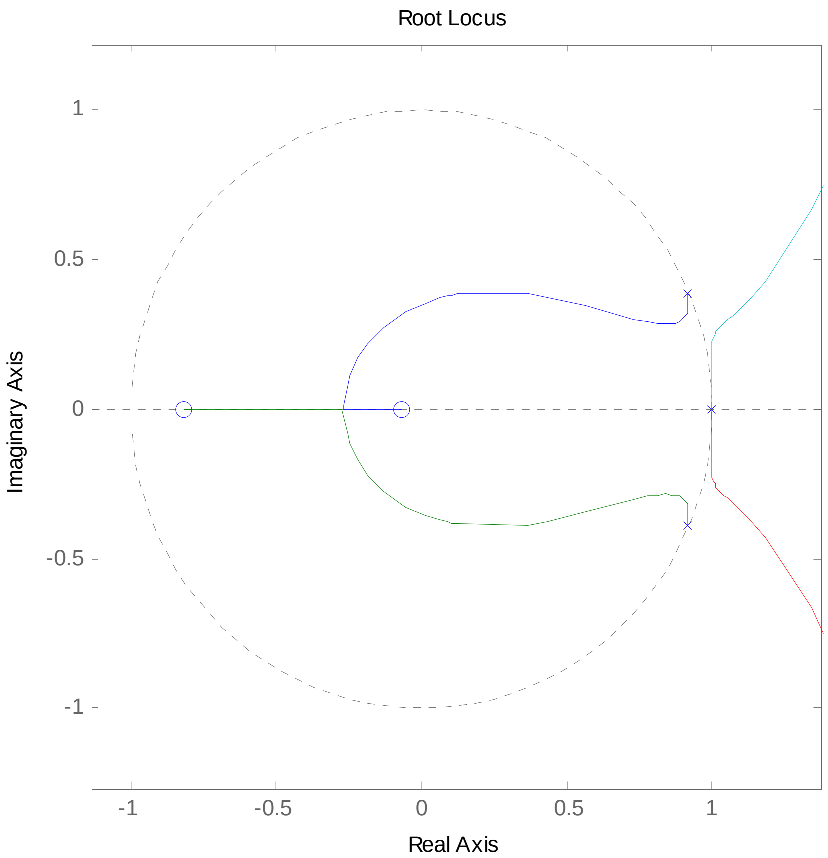


LGR Contínuo



LGR Discreto





Projeto do Observador de Estados (Estimador)

$$\hat{x}(k) - estimativa\ corrente\ (usa\ inclusive\ y(k))$$

$$\overline{x}(k) - estimativa\ \ predita\ (usa\ até\ y(k-1))$$

$$Realimentação\ de\ Estados\begin{cases} u(k) = -K\hat{x}(k) \\ u(k) = -K\overline{x}(k) \end{cases}$$

Estimador baseado na predição

$$\overline{x}(k+1) = \Phi\overline{x}(k) + \Gamma u(k)$$