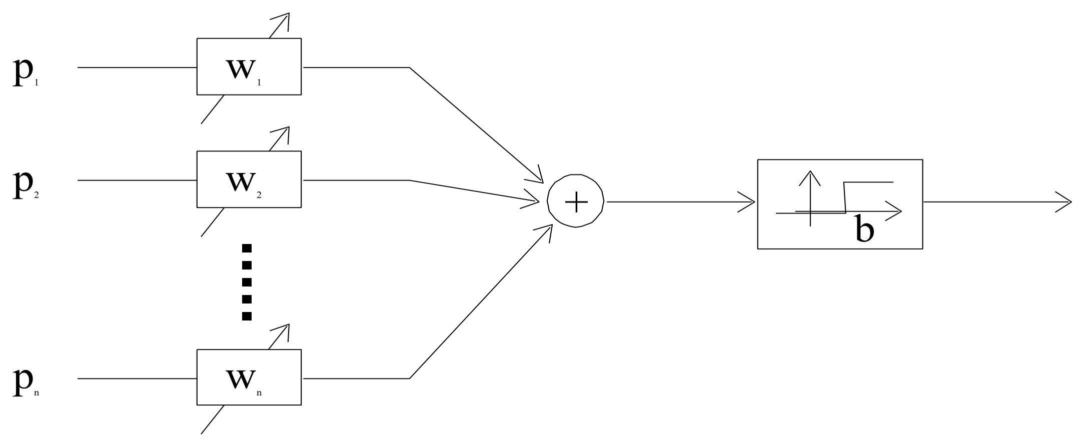
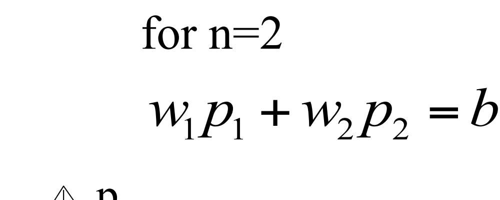
The McCulloch Neuron (1943)

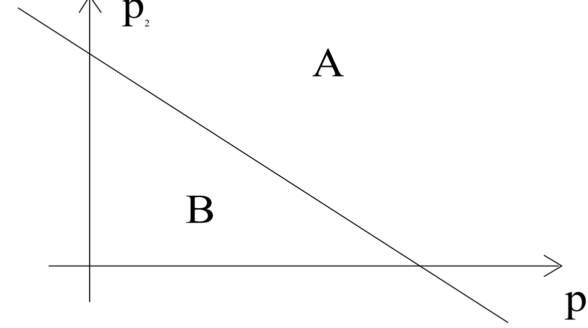


$$a = g\left(\sum_{i=1}^{n} w_i p_i - b\right) = g(\mathbf{w}^t \mathbf{p} - b) \rightarrow a \in [0;1]$$

for n=2



$$w_1 p_1 + w_2 p_2 = b$$

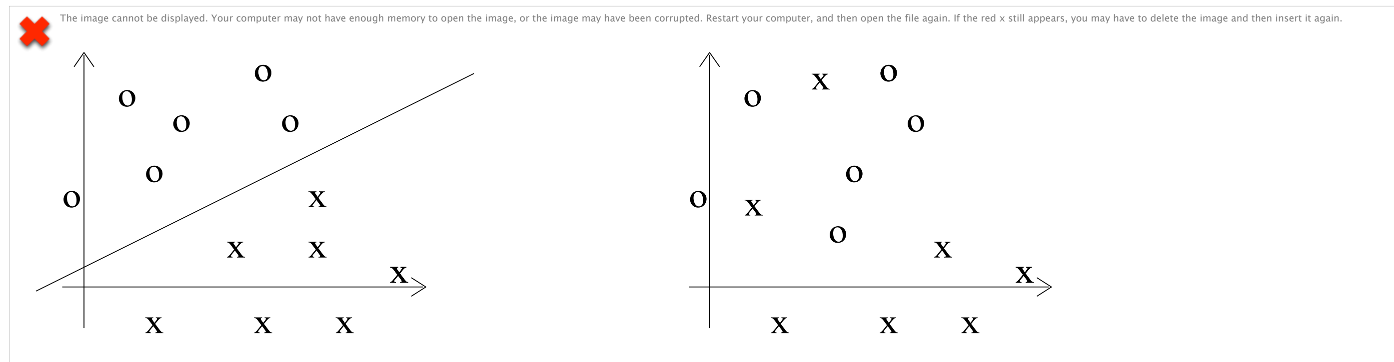


g = step function

The euclidian space $\Re^n$ is divided in two regions A and B

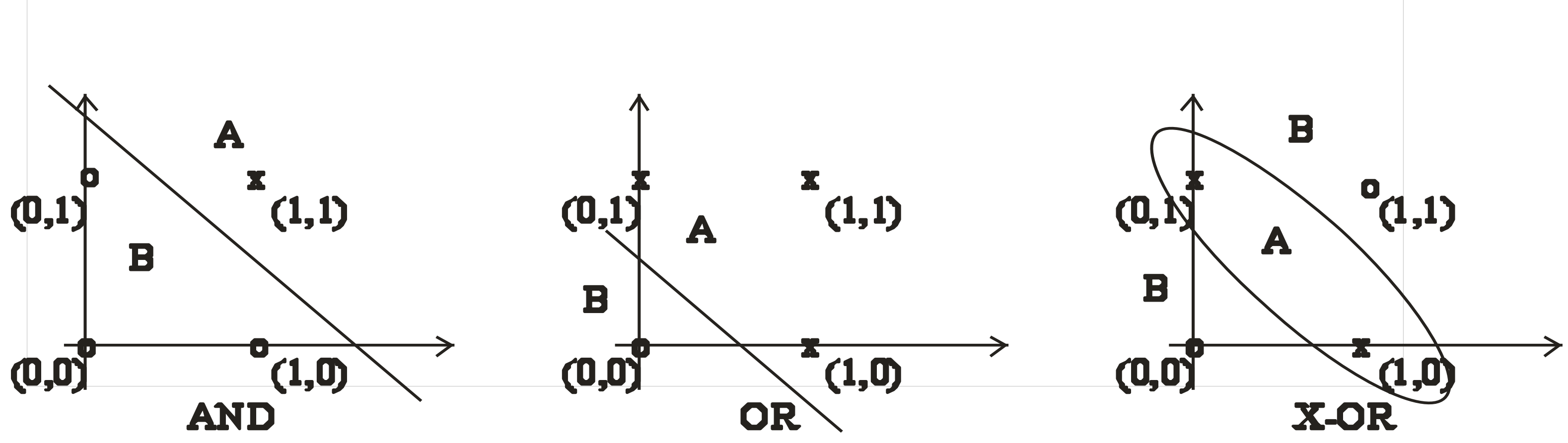
The McCulloch Neuron

– as patterns classifier



Linearly separable collections

Linearly dependent (non-separable) collections



Some Boolean functions of two variables represented in a binary plan.

Linear and Non-Linear Classifiers

There exist $2^m = 2^{2^n}$ possible logical functions connecting n inputs to one binary output.

n	# of binray patterns	# of logical functions	# linearly separable	% linearly separable
1	2	4	4	100
2	4	16	14	87,5
3	8	256	104	40,6
4	16	65536	1.772	2,9
5	32	$4{,}3 \times 10^{9}$	94.572	$2{,}2 \times 10^{-3}$
6	64	$1{,}8 \times 10^{19}$	5.028.134	$3{,}1 \times 10^{-13}$

The logical functions of one variable:

$A, \ \bar{A}, \ 0, \ 1$

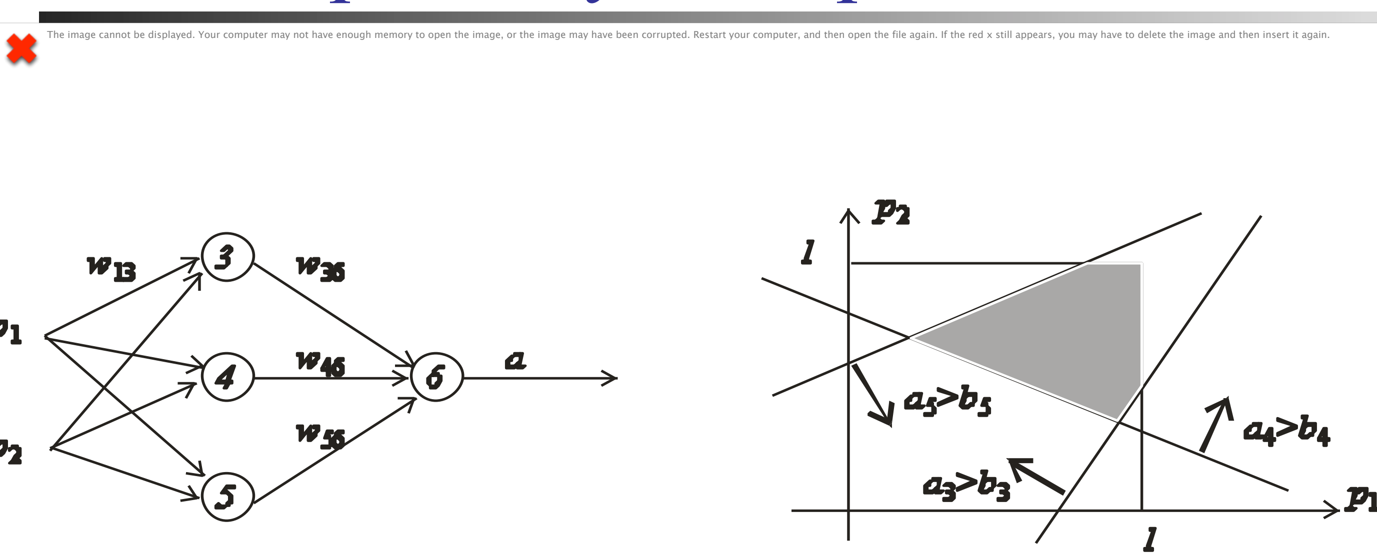
The logical functions of two variables:

$A, \ B, \bar{A}, \ \bar{B}, \ 0, \ 1$

$A \vee B, A \wedge B, \bar{A} \vee B, \bar{A} \wedge B,$

$A \vee \bar{B}, A \wedge \bar{B}, \bar{A} \vee \bar{B}, \bar{A} \wedge \bar{B}, \ A \oplus B, \overline{A \oplus B}$

Two Step Binary Perceptron



The neuron 6 implements a logical AND function by choosing

$$b_6 = \sum_{i=3}^{5} w_{i6} \, .$$

For example:

$$w_{36} = w_{46} = w_{56} = \frac{1}{3}; \quad b_6 = 1 \quad \Rightarrow a_6 = 1 \; if \;\; and \;\; only \;\; if \quad a_3 = a_4 = a_5 = 1$$

Three Step Binary Perceptron

w_{13}, w_{39}, w_{49}, w_{59}

p_1, p_2 → nodes 3, 4, 5, 6, 7, 8 → nodes 9, 10 → (A, B) → node 10 → a_{11}

$$a_{11} = A \wedge \overline{B}$$

Neurons and Artificial Neural Networks

- Micro-structure

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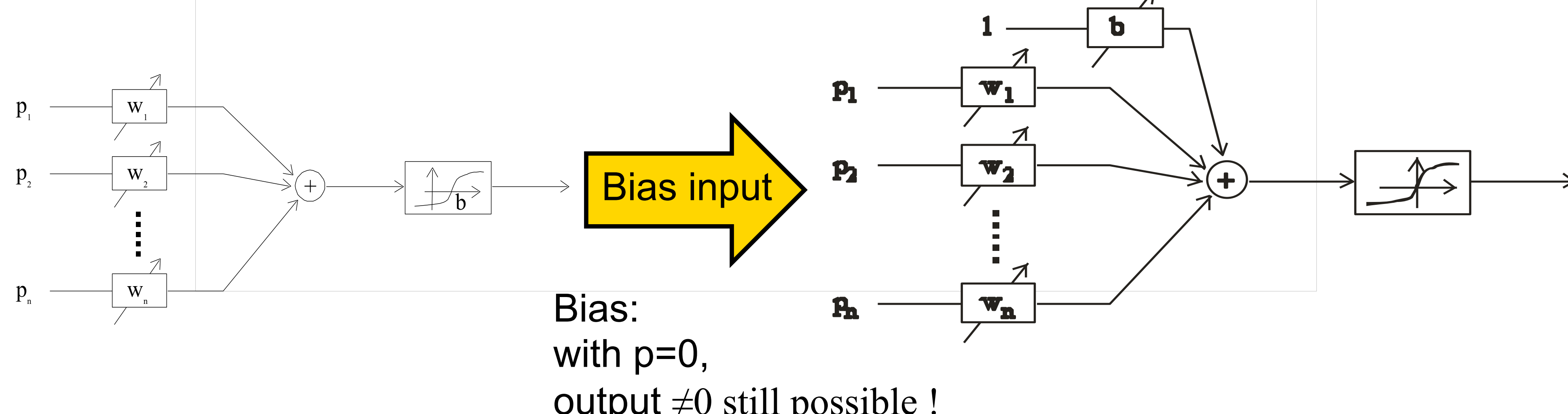
  characteristics of each neuron in the network
- Meso-Structure

  organization of the network
- Macro-Structure

  association of networks, eventually with some analytical processing approach for complex problems



Typical activation functions

Linear	$f(s) = s$	Hopfield BSB	purelin	$f(s)$, s
Signal	$f(s) = \begin{cases} +1 \;\; se \;\; s \geq 0 \\ -1 \;\; se \;\; s < 0 \end{cases}$	Perceptron	hardlims	$f(s)$, 1, -1, s
Step	$f(s) = \begin{cases} +1 \; se \; s \geq 0 \\ 0 \; se \; s < 0 \end{cases}$	Perceptron BAM	hardlim	$f(s)$, 1, s
Hopfield/ BAM	$f(s) = \begin{cases} +1 \; se \; s > 0 \\ -1 \; se \; s < 0 \\ unchanged \;\; if \;\; s = 0 \end{cases}$	Hopfield BAM		$f(s)$, 1, -1, s

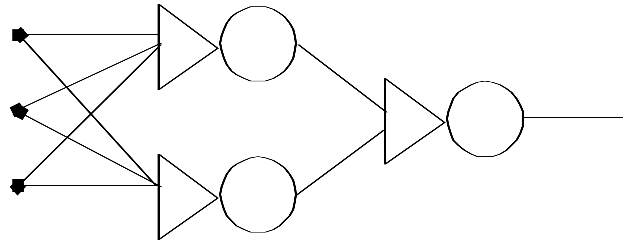
Typical activation functions

BSB or Logical Threshold	$f(s) = \begin{cases} -K & se\ s \leq -K \\ s & se\ -K < s < +K \\ +K & se\ s \geq +K \end{cases}$	BSB	satlin satlins	$f(s)$, K, $-K$, s
Logístics	$f(s) = \dfrac{1}{1+e^{-s}}$	Perceptron Hopfield BAM, BSB	logsig	$f(s)$, 1, s
Hiperbolic Tangent	$f(s) = \tanh(s) = \dfrac{1-e^{-2s}}{1+e^{-2s}}$	Perceptron Hopfield BAM, BSB	tansig	$f(s)$, 1, -1, s

Meso-Structure – Network Organization...

neurons per layer
network layers
connection type (forward, backward, lateral).

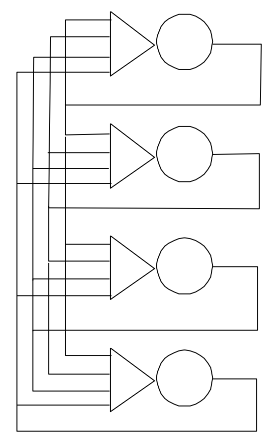
1- Multilayer *Feedforward*



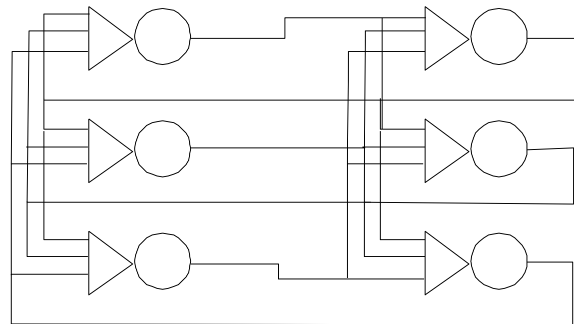
Multilayer Perceptron (MLP)

Meso-Structure – Network Organization...

2- Single Layer laterally connected (BSB (self-feedback), Hopfield)

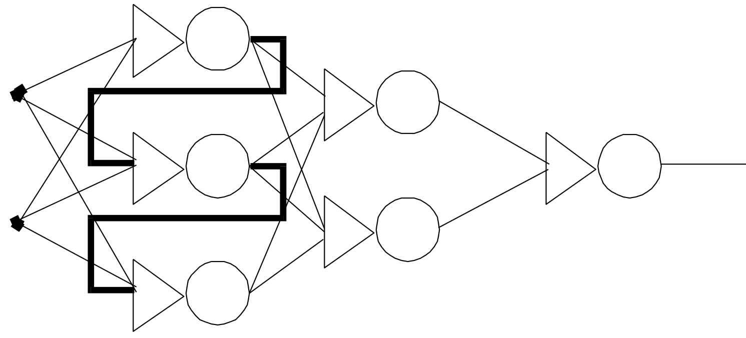


3 – Bilayers *Feedforward/Feedbackward*

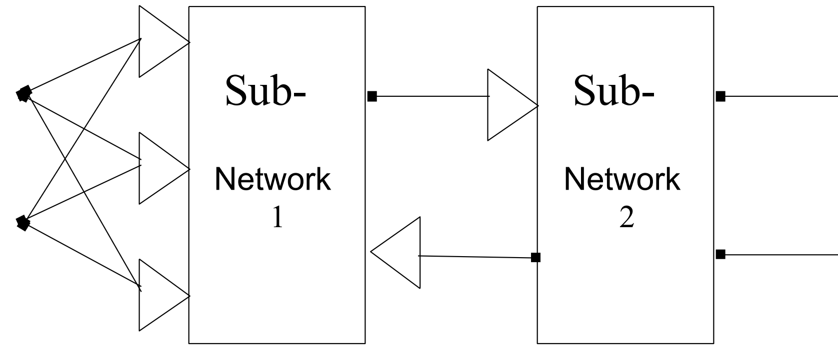


Meso-Structure – Network Organization

4 – Multilayer Cooperative/Comparative Network

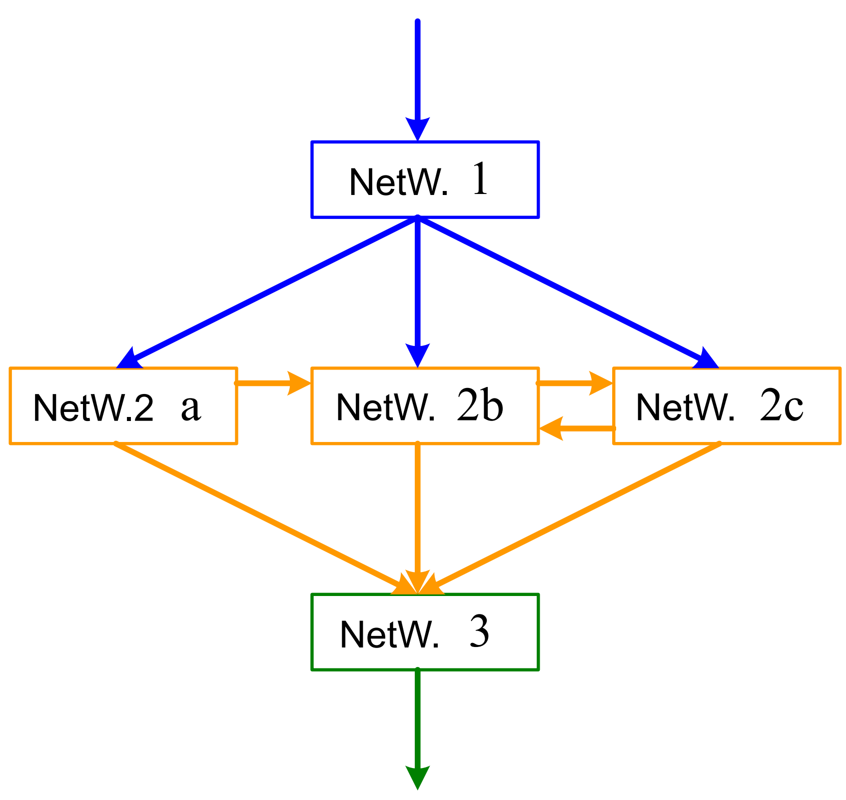


5 – Hybrid Network



Neural Macro-Structure

- # networks
- connection type
- size of networks
- degree of connectivity



Supervised Learning

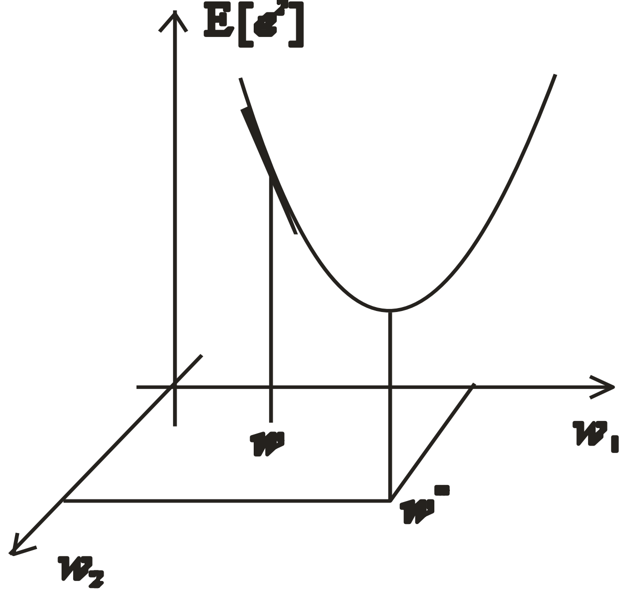
- Delta Rule → Perceptron

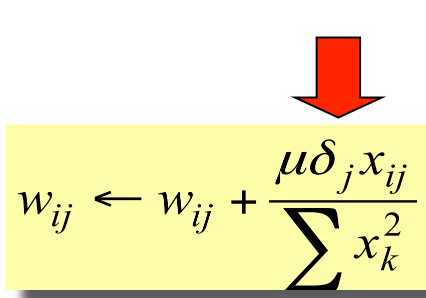
$$w \leftarrow w + \mu\delta x \qquad \delta \equiv d - y$$

μ – *learning rate*

x → y, d − y = δ

- Widrow-Hoff delta rule (LMS) → ADALINE, MADALINE
- Generalized Delta Rule





$$w_{ij} \leftarrow w_{ij} + \frac{\mu\delta_j x_{ij}}{\sum x_k^2}$$

Widrow-Hoff Delta Rule (LMS)

Delta rule → Perceptron

Perceptron – Rosenblatt, 1957

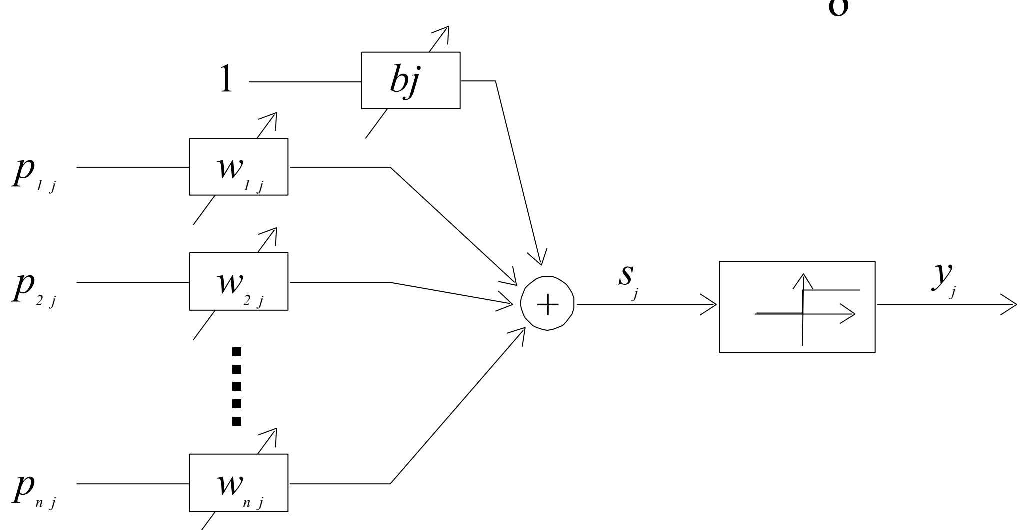
x → y

d
-
y
—
δ

Dynamics:

$$s_j = \sum_i w_{ij} p_{ij} + b_j$$

$$y_j = f(s_j) = \begin{cases} +1 & se\ s_j \geq 0 \\ 0 & se\ s_j < 0 \end{cases}$$



$$\delta_j = d_j - y_j$$

$$w_{ij} \leftarrow w_{ij} + \mu\, \delta_j x_{ij}$$ Delta Rule

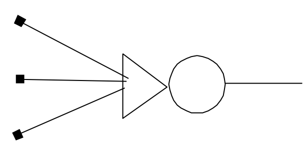
μ - learning rate

$\delta_j = 0 \rightarrow$ *the weight is not changed.*

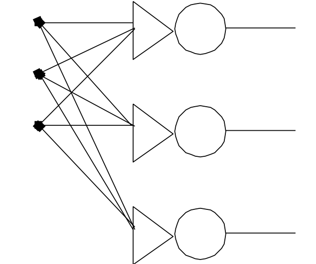
<u>Psychology Reasoning:</u>
- positive reinforcement
- negative reinforcement

ADALINE and MADALINE

Widrow & Hoff, 1960 – (Mult.) Adaptive Linear Element



$$y_j = \sum_i w_{ij} p_{ij} + b_j$$

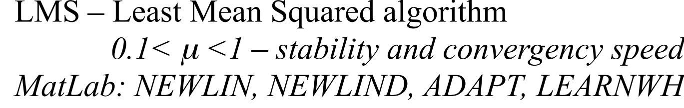


Training:

$$\varepsilon_j = d_j - s_j = d_j - \left(\sum w_{ij} p_{ij} + b_j \right)$$

$$w_{ij} \leftarrow w_{ij} + \frac{\mu \varepsilon_j x_{ij}}{\sum x_k^2}$$

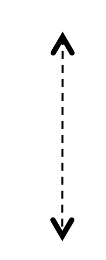
Widrow-Hoff delta rule

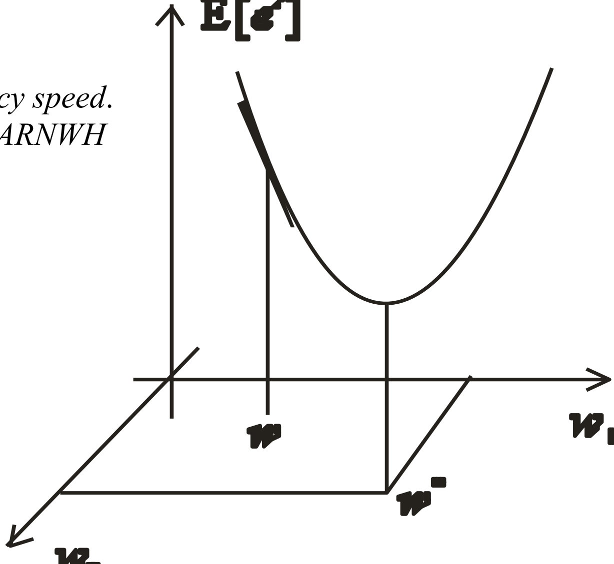


LMS – Least Mean Squared algorithm

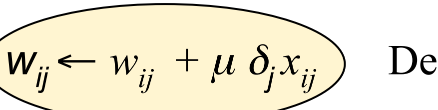
0.1< μ <1 – stability and convergency speed.

MatLab: NEWLIN, NEWLIND, ADAPT, LEARNWH





$$Obs: \ \varepsilon_j \equiv \delta_j$$



$$w_{ij} \leftarrow w_{ij} + \mu \, \delta_j x_{ij}$$

Delta Rule

LMS Algorithm

Objective: learn a function $f : \Re^n \rightarrow \Re$ from the samples ($\boldsymbol{x}_k$, d_k)

{ $\boldsymbol{x}_k$}, {d_k } and {e_k } → stationary stochastic processes

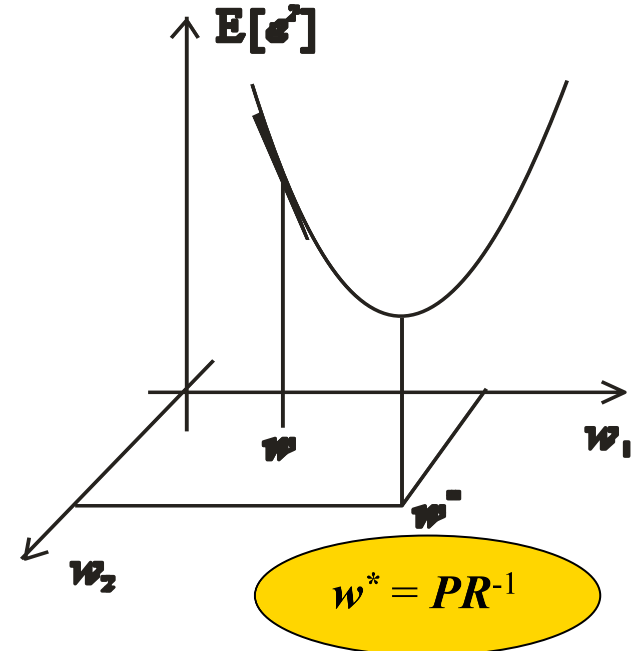
$e = d - y$ → actual stochastic error

→ Linear neuron

$$y = \sum_{i=1}^{n} x^i w^i = \mathbf{x}\mathbf{w}^t$$

Expected value

$$\begin{aligned} E[e^2] &= E[(d-y)^2] \\ &= E[(d-\boldsymbol{x}\boldsymbol{w}^t)^2] \\ &= E[d^2] - 2E[d\boldsymbol{x}]\boldsymbol{w}^t + \boldsymbol{w}E[\boldsymbol{x}^t\boldsymbol{x}]\boldsymbol{w}^t \end{aligned}$$



Optimal analytic solution of the optimization (solvelin.m)

Assuming $\boldsymbol{w}$ deterministic.
With
E [$\boldsymbol{x}^t\boldsymbol{x}$] ≡ $\boldsymbol{R}$ → autocorrelation input matrix
E [$d\boldsymbol{x}$] ≡ $\boldsymbol{P}$ → cross correlated vector

$$E[e^2] = E[d^2] - 2\boldsymbol{P}\boldsymbol{w}^t + \boldsymbol{w}\boldsymbol{R}\boldsymbol{w}^t$$

$$\mathbf{0} = 2\boldsymbol{w}^*\boldsymbol{R} - 2\boldsymbol{P}$$

(Partial derivatives equal 0 for optimal w*)

Iterative LMS Algorithm

Objective: adaptively learn a function $f : \Re^n \to \Re$ from the samples ($\boldsymbol{x}_k$, d_k)

Knowing $\boldsymbol{P}$ and $\boldsymbol{R}$, ∍ $\boldsymbol{R}^{-1}$, then for some $\boldsymbol{w}$:

$$\nabla_w \mathrm{E}[e^2] = 2\boldsymbol{wR} - 2\boldsymbol{P}$$

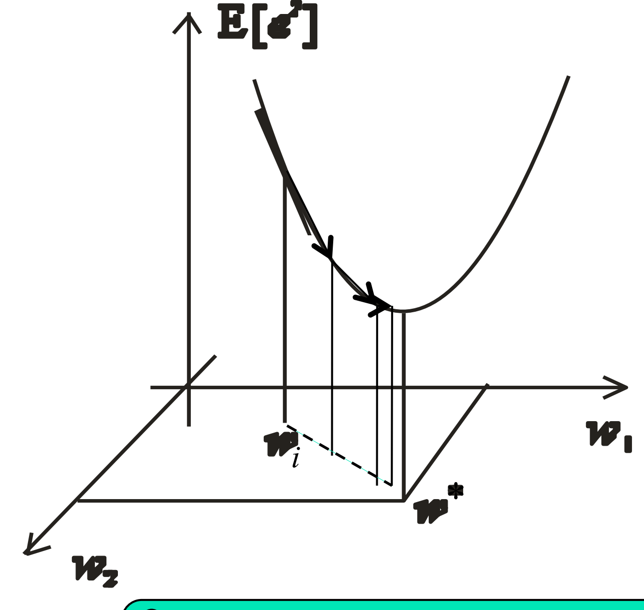
Post-multiplyting by ½ $\boldsymbol{R}^{-1}$

$$\tfrac{1}{2} \nabla_w \mathrm{E}[e^2]\, \boldsymbol{R}^{-1} = \boldsymbol{w} - \boldsymbol{P}\,\boldsymbol{R}^{-1} = \boldsymbol{w} - \boldsymbol{w}^*$$

$$\boldsymbol{w}^* = \boldsymbol{w} - \tfrac{1}{2} \nabla_w \mathrm{E}[e^2]\, \boldsymbol{R}^{-1}$$

$$\boldsymbol{w}_{k+1} = \boldsymbol{w}_k - c_k \nabla_w \mathrm{E}[e^2]\, \boldsymbol{R}^{-1}$$

($c_k = ½ \to$ Newton's method)



How to, ***cautiously*** find new (better) values for w_i , the free parameters ?

LMS Hypothesis:

$$\mathrm{E}[e^2_{k+1} | e^2_0, e^2_1, \ldots e^2_k] = e^2_k$$

Iterative LMS Algorithm...

assuming $\boldsymbol{R} = \boldsymbol{I}$ $\rightarrow$ estimated ***steppest decent algorithm***:

$$\boldsymbol{w}_{k+1} = \boldsymbol{w}_k - c_k \nabla_w e^2_{\mathrm{k}}$$

Gradient of e^2_{k} with respect to $\boldsymbol{w}$

$$\nabla_w e^2_{\mathrm{k}} = \left[\frac{\partial e_k^2}{\partial w_1}, \quad \cdots \quad \frac{\partial e_k^2}{\partial w_n}\right]$$

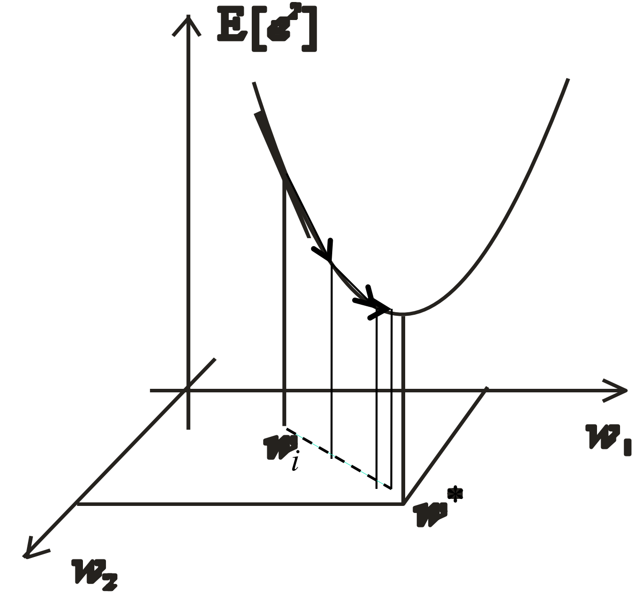
$$= \left[\frac{\partial (d_k - y_k)^2}{\partial w_1}, \quad \cdots \quad \frac{\partial (d_k - y_k)^2}{\partial w_n}\right]$$

$$= \left[-2(d_k - y_k)\frac{\partial y_k}{\partial w_1}, \quad \cdots \quad -2(d_k - y_k)\frac{\partial y_k}{\partial w_n}\right]$$

$$= -2e_k\left[\frac{\partial y_k}{\partial w_1}, \quad \cdots \quad \frac{\partial y_k}{\partial w_n}\right]$$

$$= -2e_k\left[x_k^1, \quad \cdots \quad x_k^n\right] = -2e_k\mathbf{x}_k \qquad (y_k = \mathbf{x}_k\mathbf{w}_k^t)$$

LMS algorithm reduces to $\boldsymbol{w}_{k+1} = \boldsymbol{w}_k + 2\mathrm{c}_k e_k \boldsymbol{x}_k$



$$w_{ij} \leftarrow w_{ij} + \frac{\mu\delta_j x_{ij}}{\sum x_k^2}$$

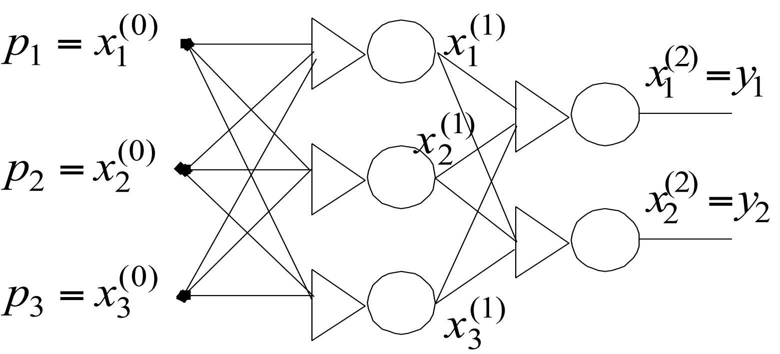
Normalization

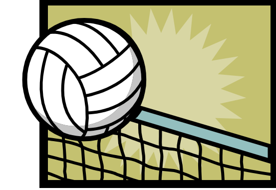
Iterative (adaptive) solution
(The optimal solution is never reached!)
MADALINE i-input, j-neuron

The Multilayer Perceptron

- The Generalized Delta Rule

Rumelhart, Hinton e Williams, PDP/MIT, 1986





Neuron Dynamics:

Processing Element (PE) j
in layer k
input i

$$s_j^{(k)} = w_{0j}^{(k)} + \sum_i w_{ij}^{(k)} x_i^{(k-1)}$$

with f (activation function) ***continuous differentiable***

$$x_j^{(k)} = f(s_j^{(k)})$$

Turning Point Question: How to find the error associated with an internal neuron??

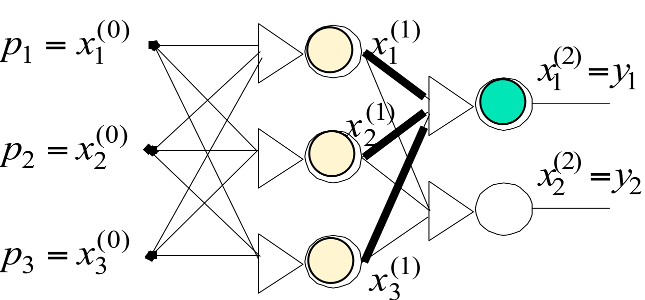
The generalized delta rule

Training

$$\varepsilon^2 = \sum_{j=1}^{m} (d_j - y_j)^2$$ - quadratic error

$$\mathbf{w}_j^{(k)} = (w_{oj}^{(k)}, w_{1j}^{(k)}, \dots, w_{mj}^{(k)})$$ - weigths of PE j

$$\mathbf{x}_j^{(k-1)} = (1, x_{1j}^{(k-1)}, \dots, x_{nj}^{(k-1)})$$ - input vector of PE j



With $s_j^{(k)} = \mathbf{w}_j^{(k)} \mathbf{x}_j^{(k-1)} \rightarrow \dfrac{\partial s_j^{(k)}}{\partial \mathbf{w}_j^{(k)}} = \mathbf{x}_j^{(k-1)}$

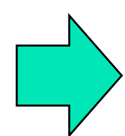
Instantaneous gradient:

$$\nabla_j^{(k)} = \frac{\partial \varepsilon^2}{\partial \mathbf{w}_j^{(k)}} = \left[\frac{\partial \varepsilon^2}{\partial w_{0j}^{(k)}}, \quad \frac{\partial \varepsilon^2}{\partial w_{1j}^{(k)}}, \quad \cdots \quad \frac{\partial \varepsilon^2}{\partial w_{mj}^{(k)}} \right]$$

$$\nabla_j^{(k)} = \frac{\partial \varepsilon^2}{\partial \mathbf{w}_j^{(k)}} = \frac{\partial \varepsilon^2}{\partial s_j^{(k)}} \frac{\partial s_j^{(k)}}{\partial \mathbf{w}_j^{(k)}}$$

so $\nabla_j^{(k)} = \dfrac{\partial \varepsilon^2}{\partial \mathbf{w}_j^{(k)}} = \dfrac{\partial \varepsilon^2}{\partial s_j^{(k)}} \mathbf{x}_j^{(k-1)}$

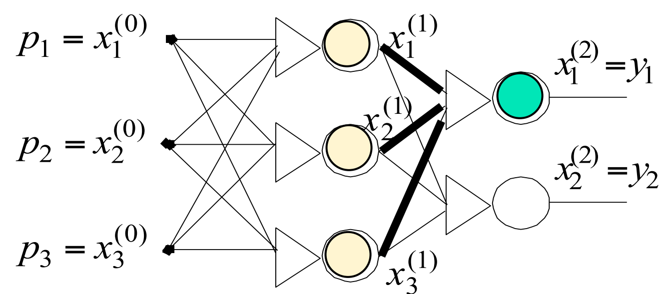
Defining the ***quadratic derivative error*** as $\delta_j^{(k)} = -\dfrac{1}{2} \dfrac{\partial \varepsilon^2}{\partial s_j^{(k)}}$



$$\nabla_j^{(k)} = -2 \delta_j^{(k)} \mathbf{x}_j^{(k-1)}$$

Gradient of the error with respect to the weights as function of the ***former layer*** signals!!

The generalized delta rule...



For the ***output layer***, the quadratic derivative error is:

$$\delta_j^{(k)} = -\frac{1}{2}\frac{\partial \sum_{i=1}^{N_k}(d_i - y_i)^2}{\partial s_j^{(k)}} = -\frac{1}{2}\frac{\partial \sum_{i=1}^{N_k}(d_i - f(s_i^{(k)}))^2}{\partial s_j^{(k)}}$$

The partial derivatives are 0 for $i \neq j$

$$\delta_j^{(k)} = -\frac{1}{2}\frac{\partial (d_j - f(s_j^{(k)}))^2}{\partial s_j^{(k)}} = -(d_j - f(s_j^{(k)}))\frac{\partial (d_j - f(s_j^{(k)}))}{\partial s_j^{(k)}} = (d_j - x_j^{(k)})f'(s_j^{(k)})$$

The output error associated with PE_j, in the last layer:

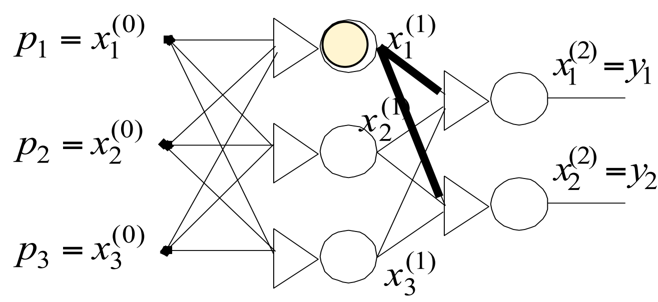
$$\varepsilon_j^{(k)} = d_j - x_j^{(k)} = d_j - y_j$$

Giving:

$$\delta_j^{(k)} = \varepsilon_j^{(k)} . f'(s_j^{(k)})$$

Remember,
"activation function, *f, **continuous differentiable***"

The generalized delta rule...



For a ***hidden layer*** *k*, the quadratic derivative error can be calculated using the linear outputs of layer *k+1*:

$$\delta_j^{(k)} = -\frac{1}{2}\frac{\partial \varepsilon^2}{\partial s_j^{(k)}} = -\frac{1}{2}\sum_{i=1}^{N_{k+1}}\left(\frac{\partial \varepsilon^2}{\partial s_i^{(k+1)}}\frac{\partial s_i^{(k+1)}}{\partial s_j^{(k)}}\right) \qquad \textit{(Chain Rule)}$$

$$= \sum_{i=1}^{N_{k+1}}\left(\left(-\frac{1}{2}\frac{\partial \varepsilon^2}{\partial s_i^{(k+1)}}\right)\frac{\partial s_i^{(k+1)}}{\partial s_j^{(k)}}\right) = \sum_{i=1}^{N_{k+1}}\left(\delta_i^{(k+1)}\frac{\partial s_i^{(k+1)}}{\partial s_j^{(k)}}\right)$$

Taking into account that $s_j^{(k)} = w_{0j}^{(k)} + \sum_{i=1}^{N_k} w_{ij}^{(k)} x_i^{(k-1)}$

$$\delta_j^{(k)} = \sum_{i=1}^{N_{k+1}}\left(\delta_i^{(k+1)}\frac{\partial}{\partial s_j^{(k)}}\left(w_{0i}^{(k+1)} + \sum_{l=1}^{N_k} w_{li}^{(k+1)} f\left(s_l^{(k)}\right)\right)\right)$$

$$\delta_j^{(k)} = \sum_{i=1}^{N_{k+1}}\left(\delta_i^{(k+1)}\sum_{l=1}^{N_k} w_{li}^{(k+1)}\frac{\partial}{\partial s_j^{(k)}} f\left(s_l^{(k)}\right)\right)$$

considering

$$\frac{\partial}{\partial s_j^{(k)}} f\left(s_l^{(k)}\right) = 0 \text{ if } l \neq j \text{ and that } \frac{\partial}{\partial s_j^{(k)}} f\left(s_j^{(k)}\right) = f'\left(s_j^{(k)}\right)$$

We have: $$\delta_j^{(k)} = \underbrace{\left(\sum_{i=1}^{N_{k+1}}\left(\delta_i^{(k+1)} w_{ji}^{(k+1)}\right)\right)}_{\equiv \varepsilon_j^{(k)}} . f'\left(s_j^{(k)}\right)$$

Finally, the ***quadratic derivative errror*** for a hidden layer:

$$\delta_j^{(k)} = \varepsilon_j^{(k)} . f'(s_j^{(k)})$$

The “Error Backpropagation” algorithm

1. $w_{ij}^{(k)} \leftarrow random$, initialize the network weigths

2. for (**x**,**d**), training pair, obtain y. Feedforward propagation: $\varepsilon^2 = \sum_{j=1}^{m}(d_j - y_j)^2$

3. k $\leftarrow last\ \ layer$

4. for each element j in the layer k do:

   Compute $\varepsilon_j^{(k)}$ using $\varepsilon_j^{(k)} = d_j - x_j^{(k)} = d_j - y_j$ if k is the last layer,

   $$\varepsilon_j^{(k)} = \sum_{i=1}^{N_{k+1}} \delta_i^{(k+1)} w_{ji}^{(k+1)}$$ if it is a hidden layer;

   Compute $\delta_j^{(k)} = \varepsilon_j^{(k)}.f'(s_j^{(k)})$

5. $k \leftarrow k-1$ if k > 0 go to step 4, else continue.

6. $\mathbf{w}_j^{(k)}(n+1) = \mathbf{w}_j^{(k)}(n) + 2\mu\delta_i^{(k)}\mathbf{x}_i^{(k)}$

7. For the next training pair go to step 2.

The Backpropagation Algorithm ***in practice***

1 – In the standard form BP is ***very slow***.

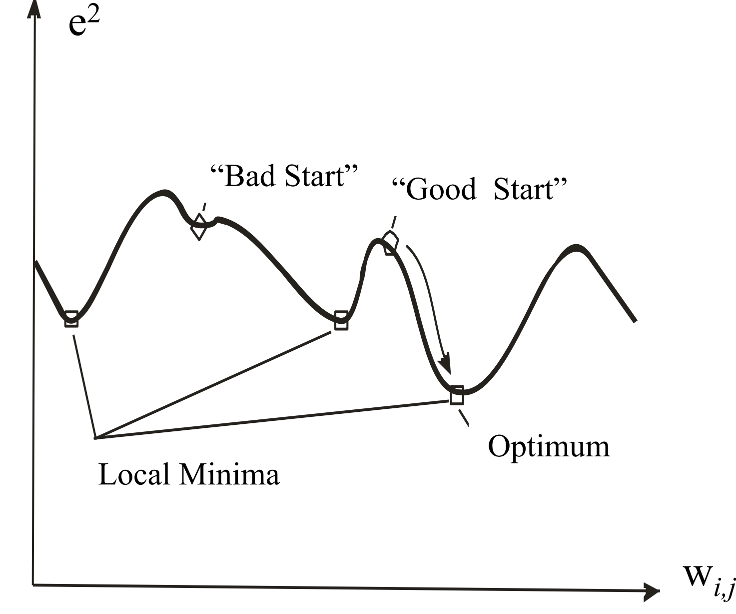
2 – BP Pathologies: ***paralysis*** in regions of small gradient.

3 – ***Initial conditions*** can lead to local minima.

4 – Stop conditions – number of epochs, $\Delta w_{ij} < \epsilon$

5 – BP variants
- trainbpm (with momentum)
- trainbpx (adaptive learning rate)
- ....
- trainlm (Levenberg-Marquard – *J*, Jacobian)

$$\Delta \mathbf{W}_j^{(k)} = (J^T J + \mu J)^{-1} J^T e$$



$e^2(w_{i,j})$ - Illustrative quadratic error as function of the weights

Obs: the error surface is, normally, unknown.

Steepest descent → go in the opposite direction of the ***local*** gradient ("downhill").

Computational Tools

- SNNS
- MatLab
  - Neural Network Toolbox
- NeuralWorks
- Java
- C++

- Hardware Implementations of RNAs

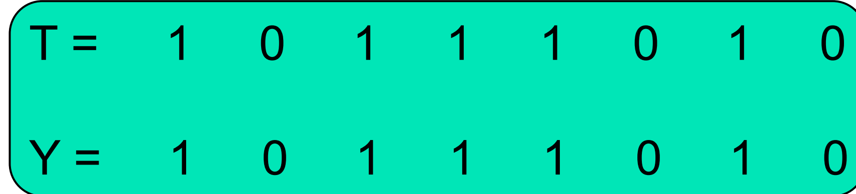
SNNS - Stuttgarter Neural Network Simulator

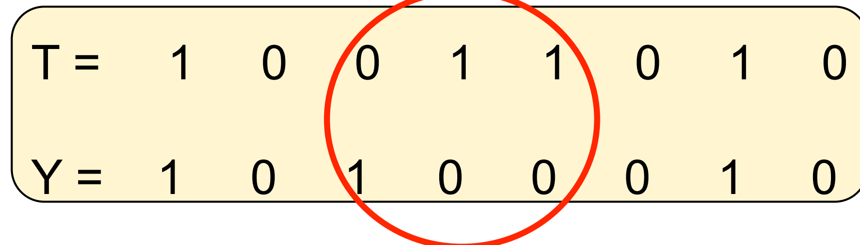
MatLab

- complete environment
  - -System Simulation
  - -Training
  - -Control

Demonstration - Perceptron

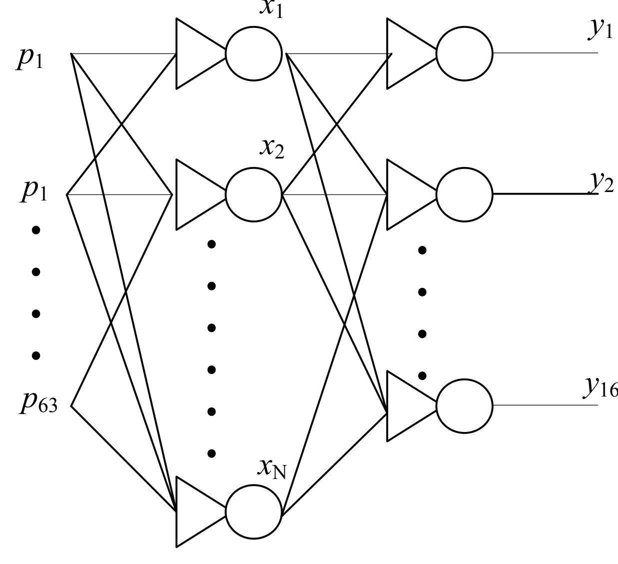
```
% Perceptron
% Training an ANN to learn to classify a non-linear problem
% Input Pattern
P=[ 0   0   0   0   1   1   1   1
    0   0   1   1   0   0   1   1
    0   1   0   1   0   1   0   1]
% Target
%T=[1 0 1 1 1 0 1 0]    % Linear separable
  T=[1 0 0 1 1 0 1 0]    %    non separable

% Try with Rosenblat's Perceptron
net=newp(P,T,'hardlim')

% train the network
net=train(net,P,T)

Y=sim(net,P)
```



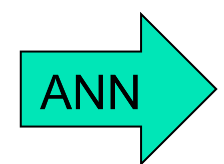


Demonstration - OCR

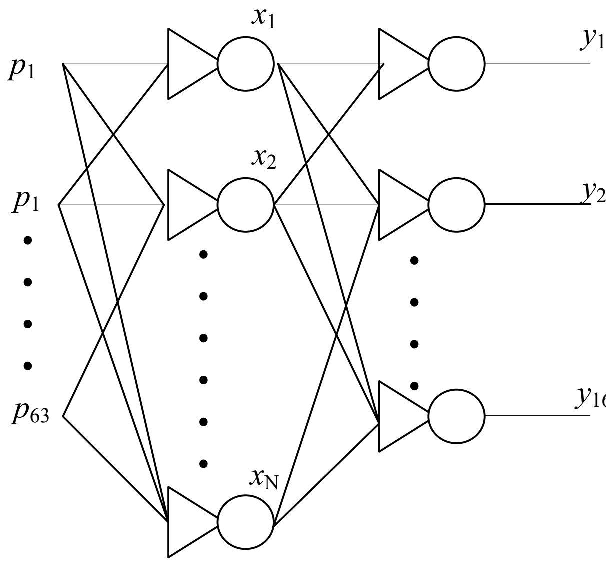
Training Vector



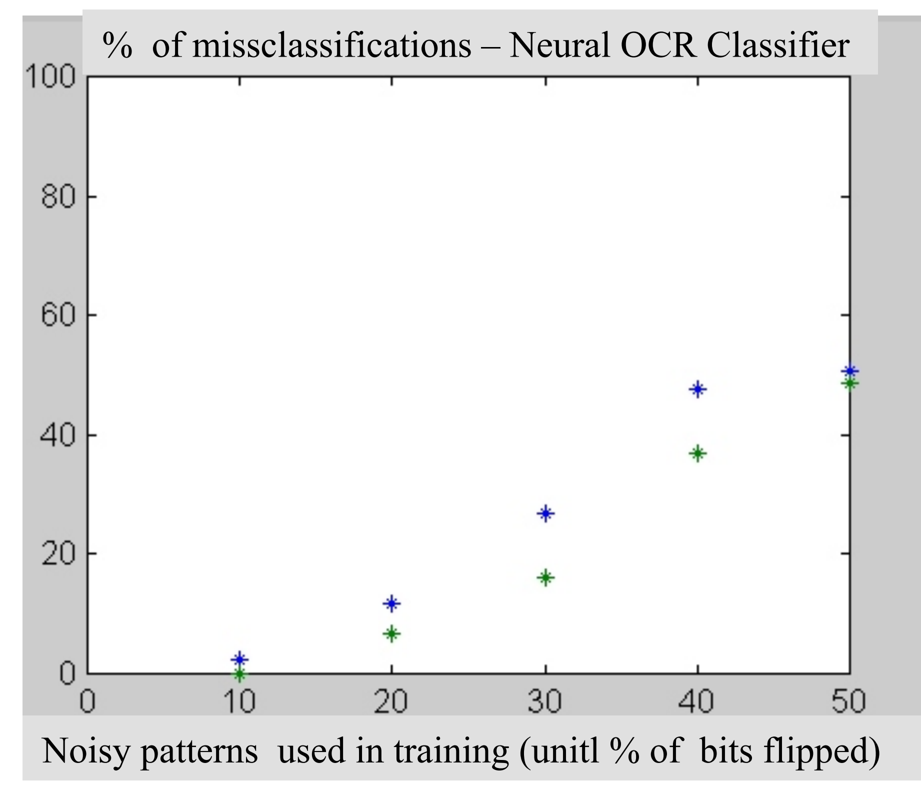
20 % Noise



Demonstration – OCR...



Training with 10 x (0,10,20,30,40,50) % noise



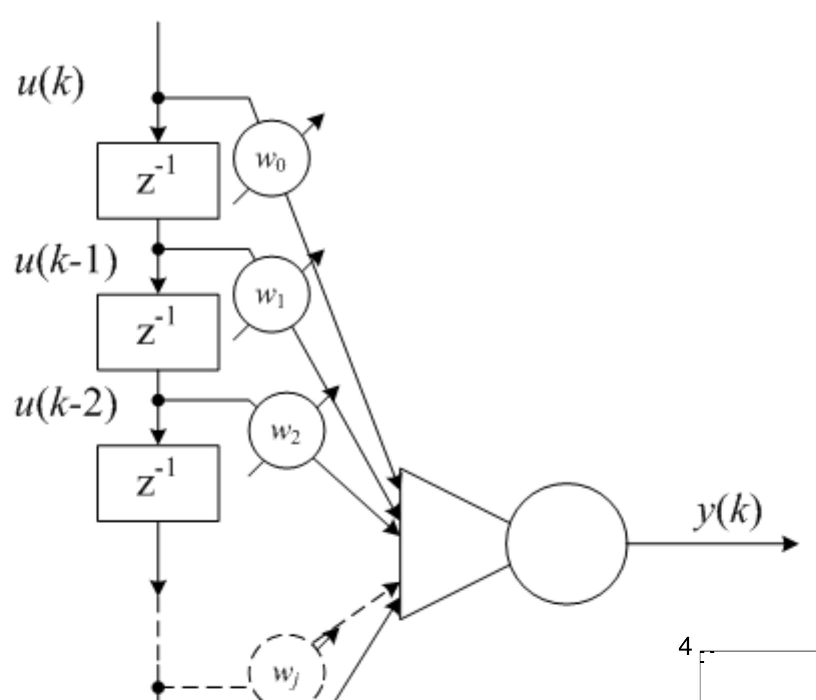
* - error without noisy training patterns
* - error using noisy training patterns

With Some Noisy Training Pattern
→ Learns how to treat "any" noise

Demonstration – LMS, ADALINE, FIR

$$y(k) = w_0 u(k) + w_1 u(k-1) + w_2 u(k-2) + \cdots w_n u(k-n)$$

$$\frac{Y(z)}{U(z)} = w_0 + w_1 z^{-1} + w_2 z^{-2} + \cdots w_n z^{-n}$$



FIR Model (always stable, only Zeros)

Obs: IIR Model is more compact, but can be unstable!

$$g_1(0-79.9\sec) = \frac{1}{s^2 + 0.2s + 1} \qquad g_2(80-150\sec) = \frac{3}{s^2 + 2s + 1}$$

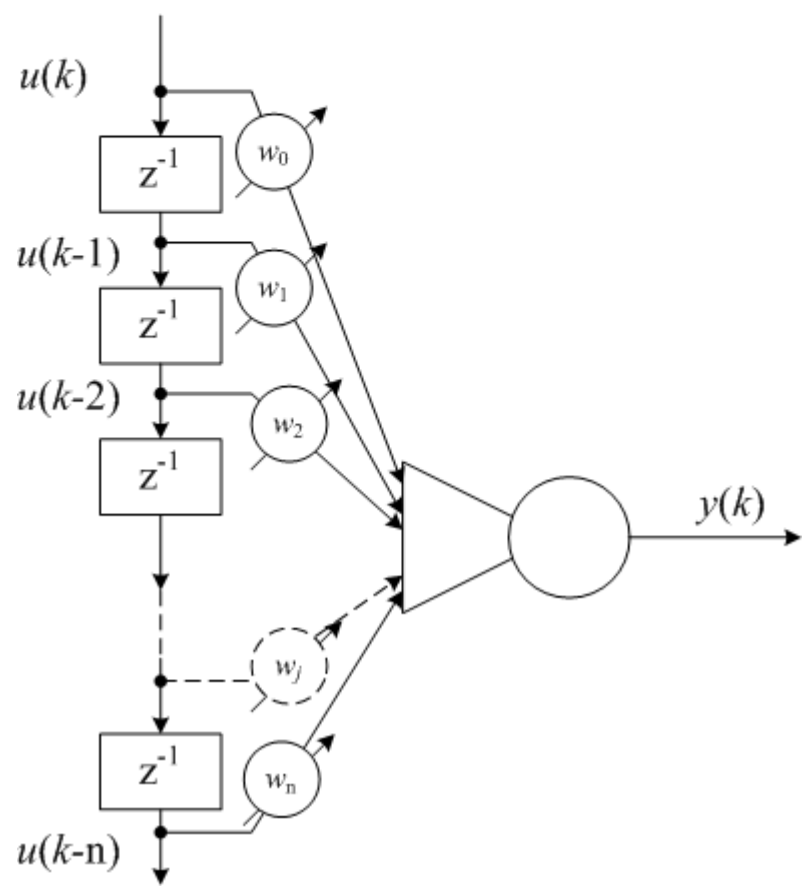
System changes at 80 sec

Sampling Time, $T_s = 0.1\sec$

4
2
0
-2
-4
0
50
100
150
sec

(TDL – Time Delay Line)

Demo – LMS, ADALINE, FIR...




```
% ADALINE - Adaptive dynamic system identification
% First sampled system - until 80 sec
g1=tf(1,[1 .2 1]), gd1=c2d(g1,.1)
 % Sytem changes dramatically - after 80 sec
g2=tf(3,[1 2 1]),gd2=c2d(g2,.1)

% Pseudo Random Binary Signal - good for identification
u=idinput(120*10,'PRBS',[0 0.01],[-1 1]);

% time vector
...
[y1,t1,x1]=lsim(gd1,u1,t1);
[y2,t2,x2]=lsim(gd2,u2,t2,x1);

% Creates new adaline nework with delayed inputs (FIR)
% Learning Rate = 0.09

net=newlin(t,y,[1 2 3 4 5 6 7 8 9 10],0.09)
[net,Y,E]=adapt(net,t,y)

% design an average transfer function
netd=newlind(t,y)
```

Demo – LMS, ADALINE, FIR...