

# Identificação de Sistemas Dinâmicos

## Filtro de Kalman

Adaptado de:

Aguirre, Cap. 9 - Introdução à Identificação de Sistemas,  
Ed. UFMG, 3ª edição

# Introdução

- Modelo Linear Discreto

$$\begin{cases} \mathbf{x}_{k+1} &= \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k \\ \mathbf{y}_{k+1} &= H_{k+1} \mathbf{x}_{k+1} + \mathbf{v}_{k+1}, \end{cases}$$

$$E[\mathbf{w}_k \mathbf{w}_k^T] = Q_k, \quad E[\mathbf{v}_k \mathbf{v}_k^T] = R_k$$

$$E[\mathbf{v}_i \mathbf{w}_j^T] = 0, \quad \forall i, j.$$

*Condições Iniciais*  $\hat{\mathbf{x}}_0, P_0$

*sequência de entrada*  $\mathbf{u}_k \in \mathfrak{R}^r$  e *medições*  $\mathbf{y}_k \in \mathfrak{R}^m$

# Função de Probabilidade

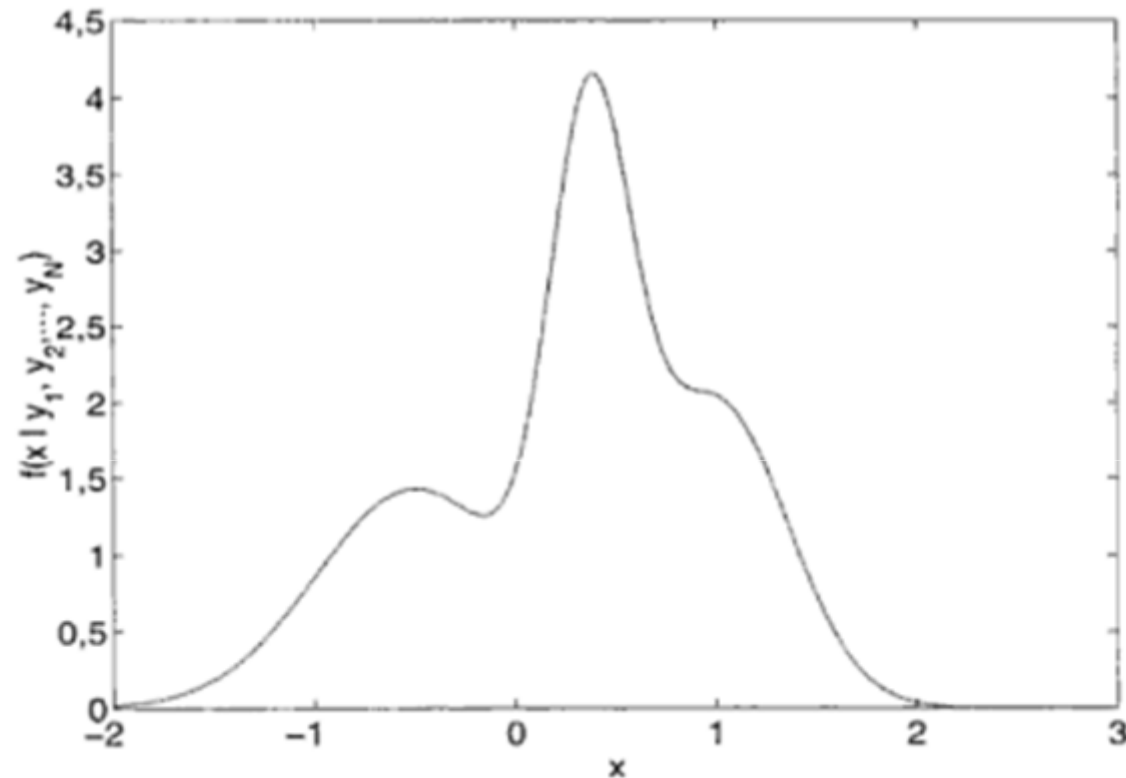
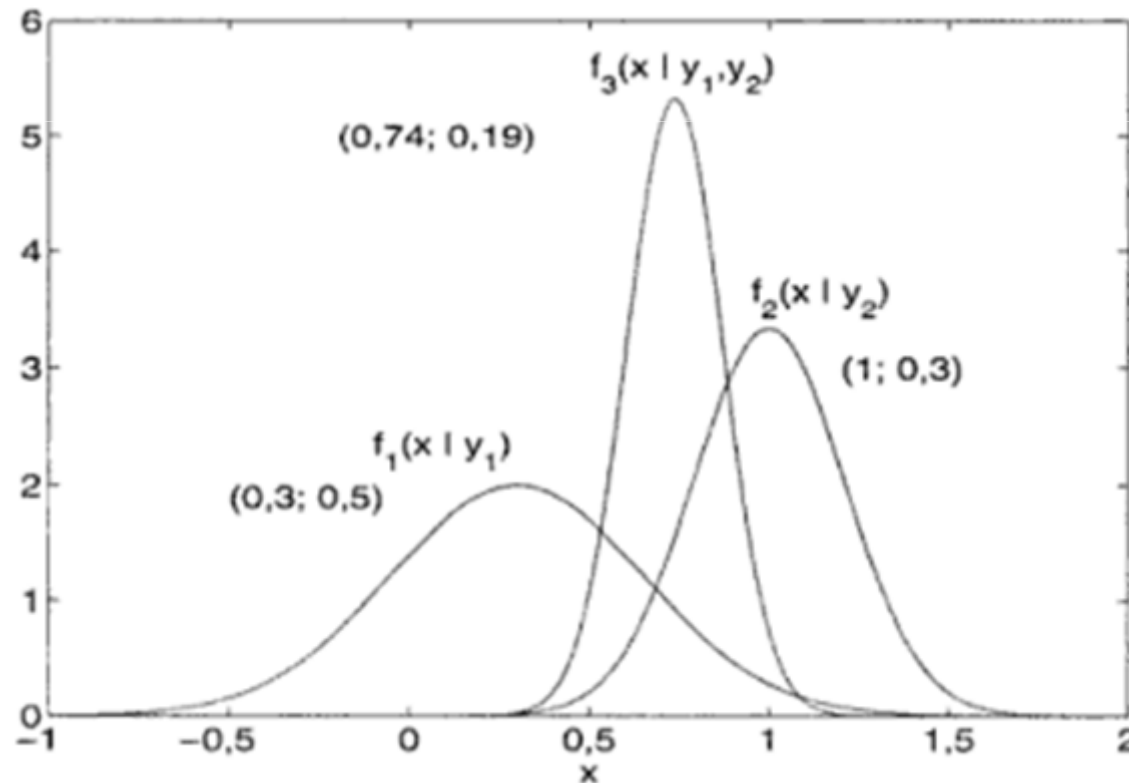


FIGURA 9.1: Função de probabilidade geral

Função de densidade de probabilidade condicional de  $x$  dadas as medições  $y_i$ ,  $i = 1, \dots, N$ . No presente caso hipotético, nota-se que valores em torno de  $x = 0,4$  são os mais prováveis. O fato de  $f(x|y_1, \dots, y_N)$  ser bastante dispersa em torno de  $x = 0,4$  indica pouca confiança nas medições.

# Função de densidade de Probabilidade Gaussiana

$$f_{\mathbf{z}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |P|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \bar{\mathbf{z}})^T P^{-1} (\mathbf{x} - \bar{\mathbf{z}}) \right\} \quad f_z(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{1}{2\sigma^2} (x - \bar{z})^2 \right\}$$



$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} y_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} y_2$$

$$\sigma^2 = \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1}$$

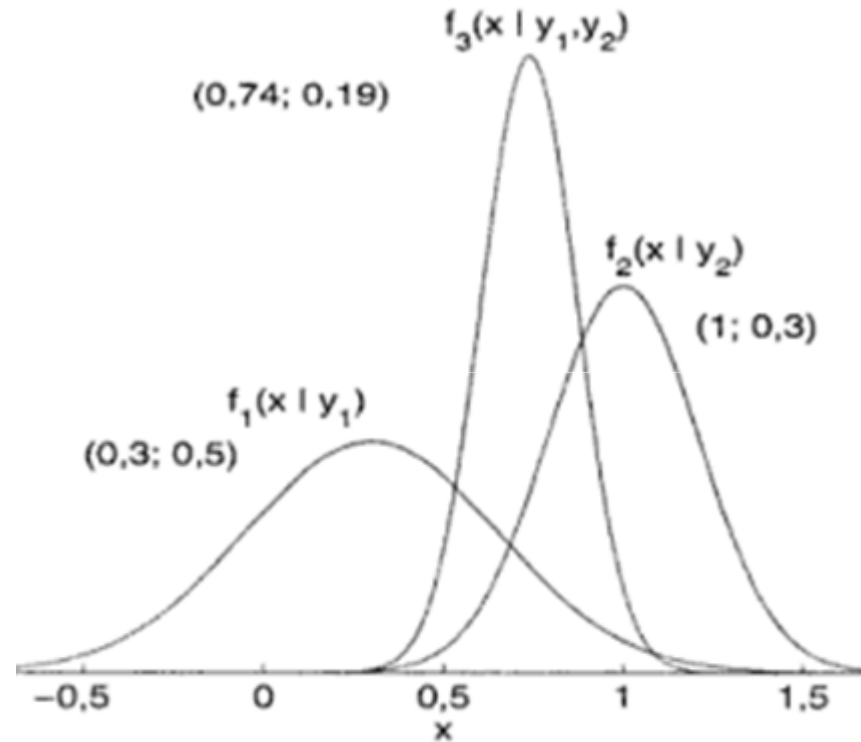
# Combinação de 2 sensores

$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} y_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} y_2$$

$$\sigma^2 = \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1}$$

$$\begin{aligned} \hat{x}_2 &= \frac{\sigma_2^2 y_1 + \sigma_1^2 y_2 + \sigma_1^2 y_1 - \sigma_1^2 y_1}{\sigma_1^2 + \sigma_2^2} \\ &= \frac{(\sigma_1^2 + \sigma_2^2) y_1 + \sigma_1^2 (y_2 - y_1)}{\sigma_1^2 + \sigma_2^2} \\ &= y_1 + K (y_2 - y_1), \end{aligned}$$

$$K = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \quad \text{e} \quad \sigma^2 = \sigma_1^2 - K \sigma_1^2$$



# Medidas de um mesmo instrumento

$$\hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[y(t_2) - \hat{x}(t_1)],$$

$$K(t_2) = \frac{\sigma^2(t_1)}{\sigma^2(t_1) + \sigma^2(t_2)}$$

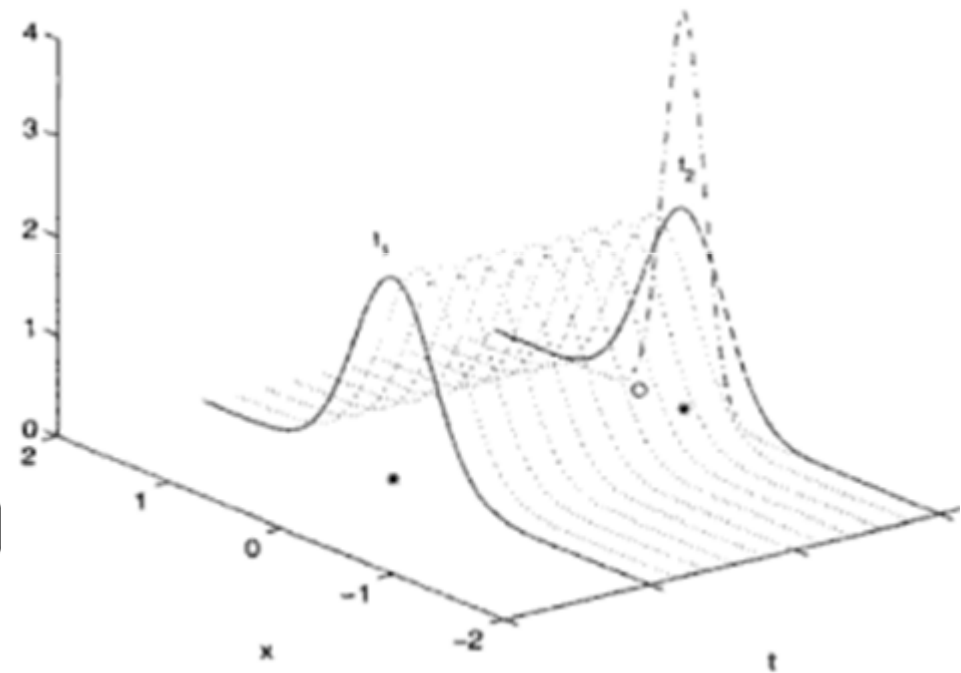
$$\sigma^2(t_2) = \sigma^2(t_1) - K(t_2)\sigma^2(t_1).$$

Nova Nomenclatura

$$\hat{x}(t_2|t_2) = \hat{x}(t_2|t_1) + K(t_2)[y(t_2) - \hat{x}(t_2|t_1)]$$

$$K(t_2) = \frac{\sigma^2(t_2|t_1)}{\sigma^2(t_2|t_1) + \sigma^2(t_2)},$$

$$\sigma^2(t_2|t_2) = \sigma^2(t_2|t_1) - K(t_2)\sigma^2(t_2|t_1).$$



# O Caso Dinâmico

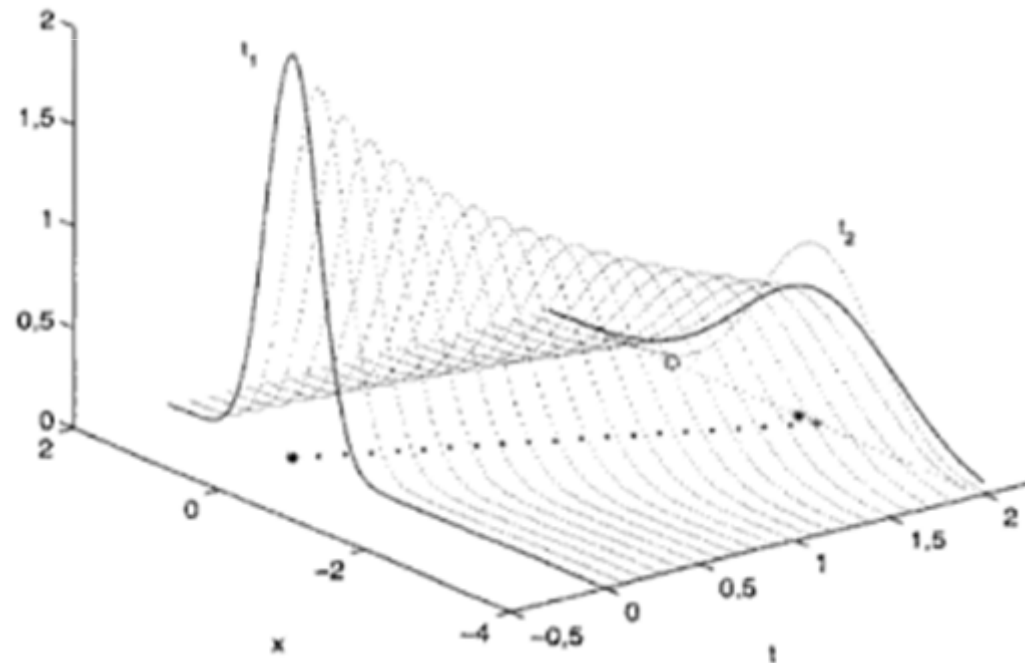
Movimento do Objeto:

e.g. Aproximação de Euler

$$\frac{x(t_2) - x(t_1)}{t_2 - t_1} \approx v + w(t_2)$$
$$x(t_2) \approx x(t_1) + (t_2 - t_1)[v + w(t_2)],$$

$$\hat{x}(t_2|t_1) = \hat{x}(t_1|t_1) + (t_2 - t_1)v$$

$$\sigma^2(t_2|t_1) = \sigma^2(t_1|t_1) + (t_2 - t_1)\sigma_w^2.$$



# O Filtro de Kalman Discreto

Lei de Movimento – propagação do vetor de estados estimado:

$$\hat{\mathbf{x}}(t_2|t_1) = \Phi(t_1)\hat{\mathbf{x}}(t_1|t_1) + \Gamma(t_1)\mathbf{u}(t_1).$$

Nova Nomenclatura

$$\hat{\mathbf{x}}_{k+1}^- = \Phi_k \hat{\mathbf{x}}_k^+ + \Gamma_k \mathbf{u}_k$$

Predição

$$\hat{\mathbf{x}}_{k+1}^+ = \hat{\mathbf{x}}_{k+1}^- + K_{k+1} [\mathbf{y}_{k+1} - H_{k+1} \hat{\mathbf{x}}_{k+1}^-]$$

Correção



# Matriz de Covariância

- Erro de estimação na propagação  $\hat{\mathbf{x}}_{k+1}^- = \hat{\mathbf{x}}_{k+1}^- - \mathbf{x}_{k+1}$

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1}^- &= [\Phi_k \hat{\mathbf{x}}_k^+ + \Gamma_k \mathbf{u}_k] - [\Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k] \\ &= \Phi_k (\hat{\mathbf{x}}_k^+ - \mathbf{x}_k) - \Upsilon_k \mathbf{w}_k \\ &= \Phi_k \tilde{\mathbf{x}}_k^+ - \Upsilon_k \mathbf{w}_k.\end{aligned}$$

Matriz de covariância de  $\hat{\mathbf{x}}_{k+1}^-$

$$\begin{aligned}P_{k+1}^- &= E[(\hat{\mathbf{x}}_{k+1}^- - E[\hat{\mathbf{x}}_{k+1}^-])(\hat{\mathbf{x}}_{k+1}^- - E[\hat{\mathbf{x}}_{k+1}^-])^T] \\ &= E[(\hat{\mathbf{x}}_{k+1}^- - \mathbf{x}_{k+1})(\hat{\mathbf{x}}_{k+1}^- - \mathbf{x}_{k+1})^T] \\ &= E[\tilde{\mathbf{x}}_{k+1}^- (\tilde{\mathbf{x}}_{k+1}^-)^T],\end{aligned}$$

...

# Equações de Filtro de Kalman Discreto

$$\left\{ \begin{array}{l} \hat{\mathbf{x}}_{k+1}^- = \Phi_k \hat{\mathbf{x}}_k^+ + \Gamma_k \mathbf{u}_k; \\ P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T; \\ K_{k+1} = P_{k+1}^- H_{k+1}^T [H_{k+1} P_{k+1}^- H_{k+1}^T + R_{k+1}]^{-1}; \\ \hat{\mathbf{x}}_{k+1}^+ = \hat{\mathbf{x}}_{k+1}^- + K_{k+1} [\mathbf{y}_{k+1} - H_{k+1} \hat{\mathbf{x}}_{k+1}^-]; \\ P_{k+1}^+ = P_{k+1}^- - K_{k+1} H_{k+1} P_{k+1}^-; \end{array} \right.$$

$$\left\{ \begin{array}{l} \hat{\mathbf{x}}_{k+1} = \Phi_k \hat{\mathbf{x}}_k + \Gamma_k \mathbf{u}_k + \Phi_k K_k [\mathbf{y}_k - H_k \hat{\mathbf{x}}_k]; \\ K_k = P_k H_k^T [H_k P_k H_k^T + R_k]^{-1}; \\ P_{k+1} = \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T, \end{array} \right.$$

# Estimação conjunto de estados e parâmetros

Matriz aumentada:

$$\begin{bmatrix} \hat{\mathbf{x}}_{k+1}^+ \\ \hat{\boldsymbol{\theta}}_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi_k & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_k^+ \\ \hat{\boldsymbol{\theta}}_k \end{bmatrix} + \begin{bmatrix} \Gamma_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_k \\ \mathbf{0} \end{bmatrix}$$
$$\hat{\mathbf{x}}_{k+1}^{*-} = \Phi_k^* \hat{\mathbf{x}}_k^{*-} + \Gamma_k^* \hat{\mathbf{u}}_k^*,$$

# O Filtro de Kalman Estendido

Modelo não-linear

$$\begin{cases} \mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k), \\ \mathbf{y}_{k+1} = h(\mathbf{x}_{k+1}) + \mathbf{v}_{k+1}, \end{cases}$$

Jacobianas:

$$Df(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$Df(\mathbf{w}) = \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \frac{\partial f_1}{\partial w_2} & \cdots & \frac{\partial f_1}{\partial w_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial w_1} & \frac{\partial f_n}{\partial w_2} & \cdots & \frac{\partial f_n}{\partial w_n} \end{bmatrix}$$

$$\left\{ \begin{aligned} \hat{\mathbf{x}}_{k+1}^- &= f(\hat{\mathbf{x}}_k^+, \mathbf{u}_k); \\ P_{k+1}^- &= Df(\mathbf{x}_k) P_k^+ Df(\hat{\mathbf{x}}_k^+)^T + Df(\mathbf{w}_k) Q_k Df(\mathbf{w}_k)^T; \\ K_{k+1} &= P_{k+1}^- Dh(\mathbf{x}_{k+1})^T [Dh(\mathbf{x}_{k+1}) Dh(\mathbf{x}_{k+1})^T + \\ &\quad + Dh(\mathbf{x}_{k+1}) R_{k+1} Dh(\mathbf{x}_{k+1})^T]^{-1}; \\ \hat{\mathbf{x}}_{k+1}^+ &= \hat{\mathbf{x}}_{k+1}^- + K_{k+1} [\mathbf{y}_{k+1} - h(\hat{\mathbf{x}}_{k+1}^-)]; \\ P_{k+1}^+ &= P_{k+1}^- - K_{k+1} Dh(\mathbf{x}_{k+1}) P_{k+1}^-, \end{aligned} \right.$$

$$Df(\mathbf{x}_k) = Df(\mathbf{x}_k)|_{\mathbf{x}_k=(\hat{\mathbf{x}}_k^+, \mathbf{u}_k, 0)}$$

$$Df(\mathbf{w}_k) = Df(\mathbf{w}_k)|_{\mathbf{x}_k=(\hat{\mathbf{x}}_k^+, \mathbf{u}_k, 0)}$$

$$Dh(\mathbf{x}_{k+1}) = Dh(\mathbf{x}_{k+1})|_{\mathbf{x}_{k+1}=\hat{\mathbf{x}}_{k+1}^-}$$

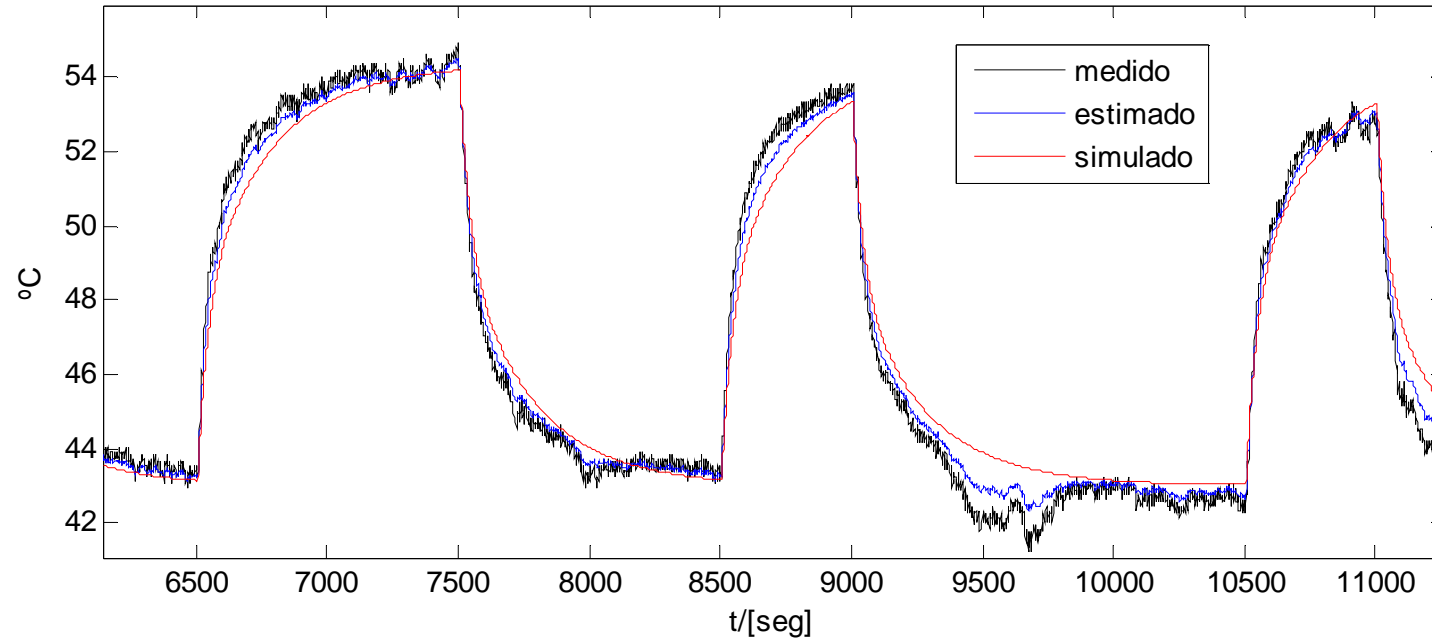
# Estimação Caixa Cinza

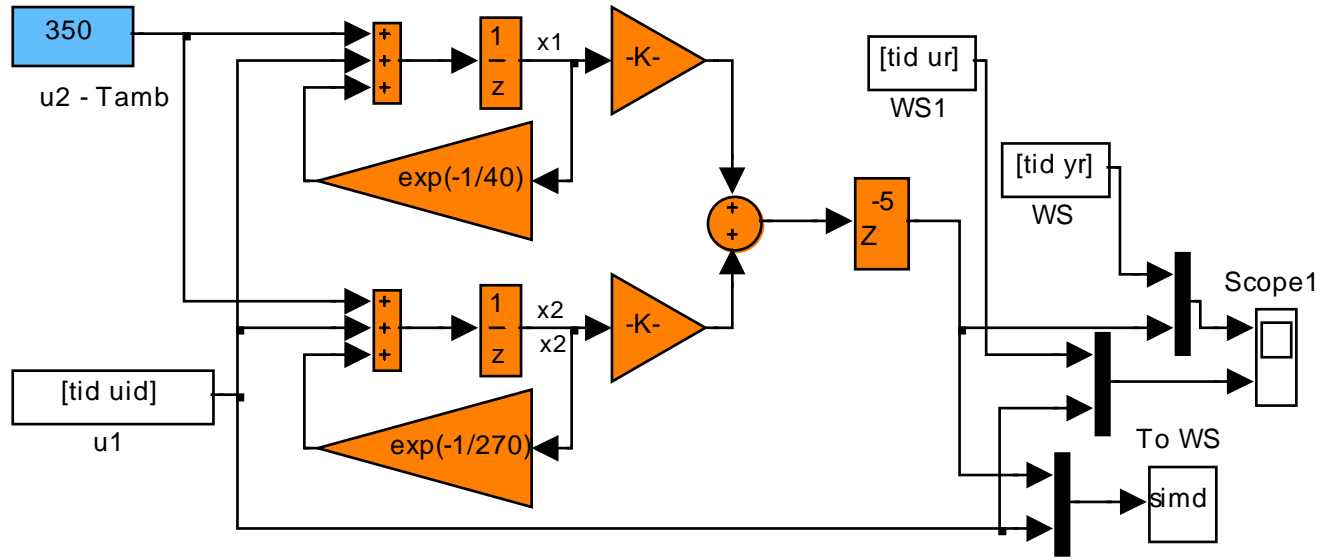
- Conhecimento à priori  
(da estrutura, via leis físicas, princípios fundamentais)
- Informação auxiliar

# Estimador MQR

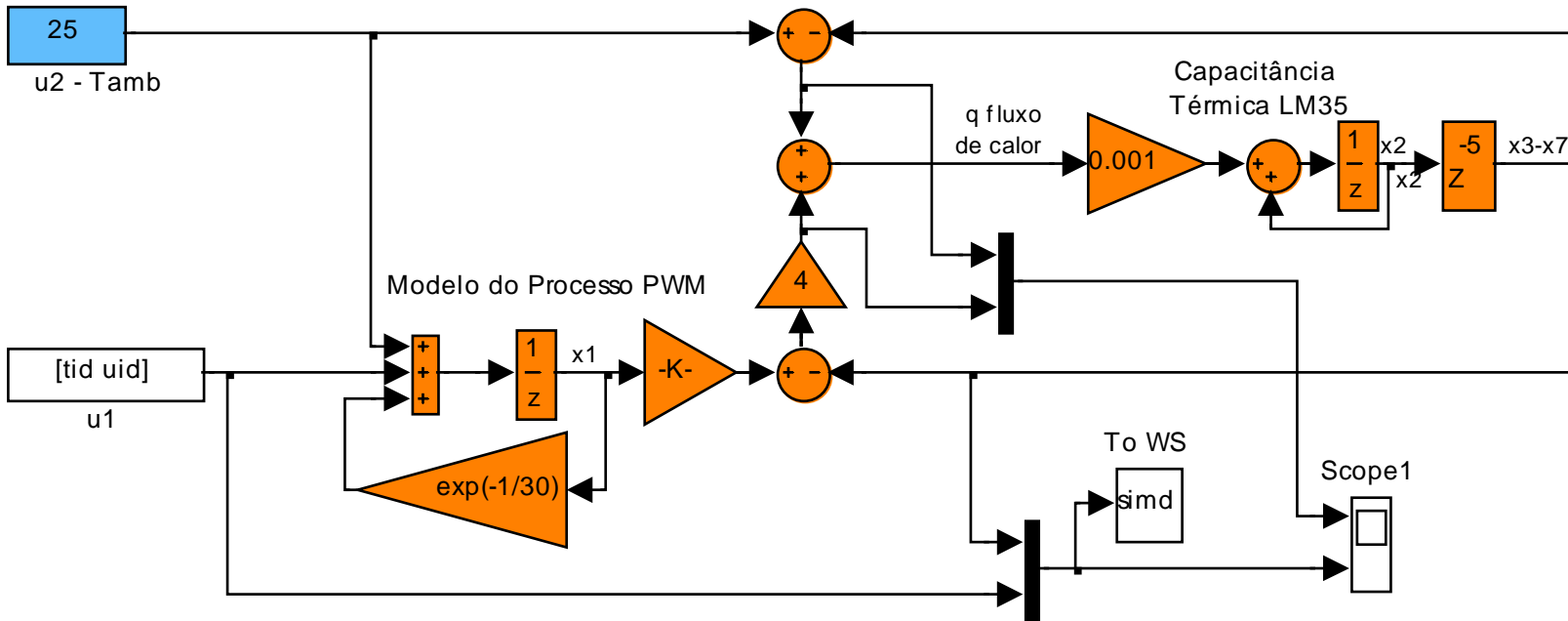
# Ex.1 – Processo Térmico

Identificação Processo Térmico - Filtro de Kalman







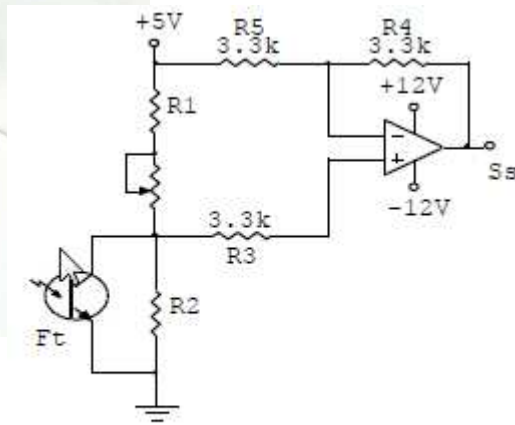
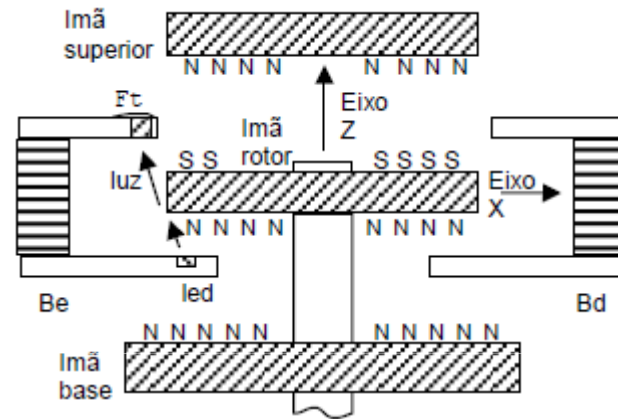
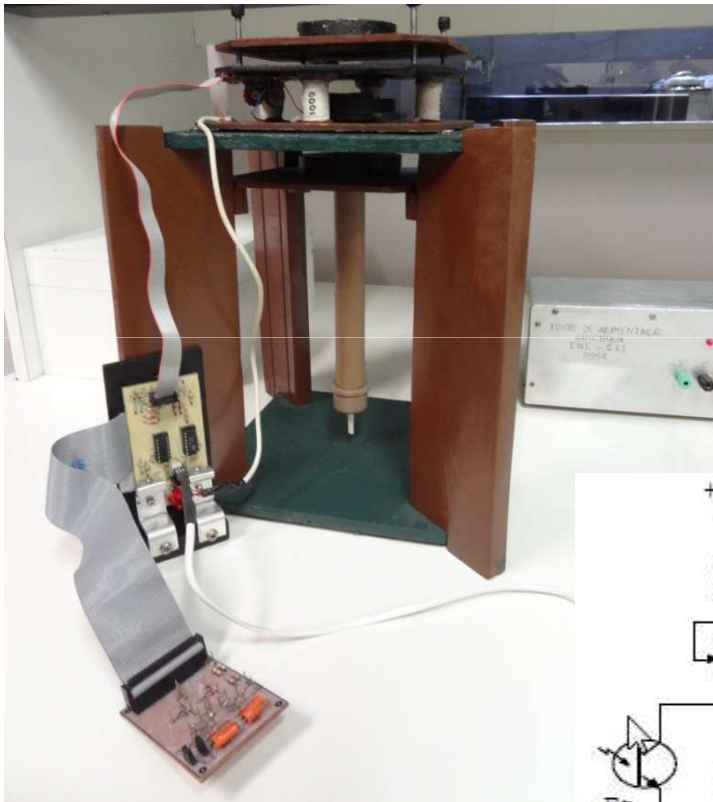


```

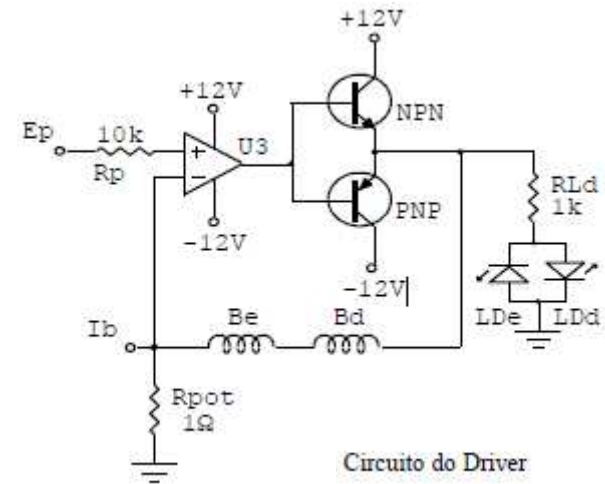
%Modelo discreto do processo térmico no espaço de estados
% x(k+1)=phi*x(k)+gama*u(k)
% y(k) = C*x(k)
phi=[exp(-1/30)           0 0 0 0 0 0
      (1-exp(-1/30))*4*0.001 1 0 0 0 0 -5*0.001
      0                   1 0 0 0 0 0
      0                   0 1 0 0 0 0
      0                   0 0 1 0 0 0
      0                   0 0 0 1 0 0
      0                   0 0 0 0 1 0]
gama=[1 1
      0 0.001
      0 0
      0 0
      0 0
      0 0]
H=[0 0 0 0 0 0 1];

```

# Ex. 2 Levitação Magnética



Circuito sensor de posição



Circuito do Driver

# Ex. 2 Levitação Magnética

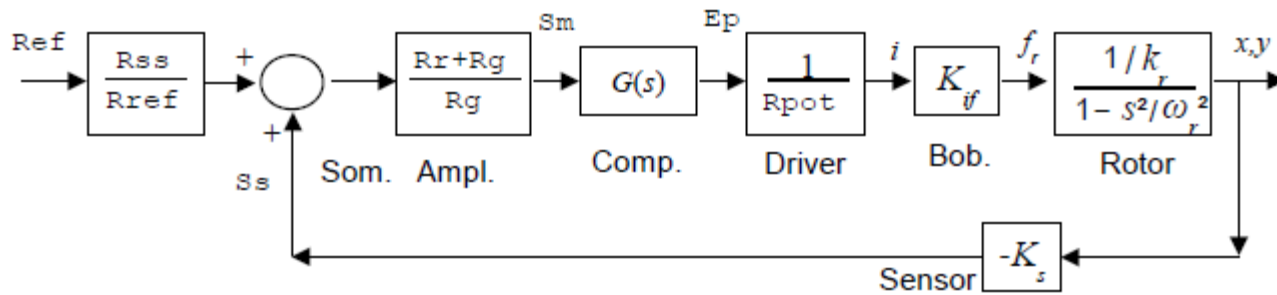
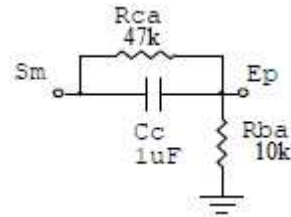


Diagrama de blocos em um dos canais

$$\frac{X(s)}{F_x(s)} = \frac{1/k_r}{1 - \frac{s^2}{\omega_r^2}}$$

Processo



Compensador em avanço

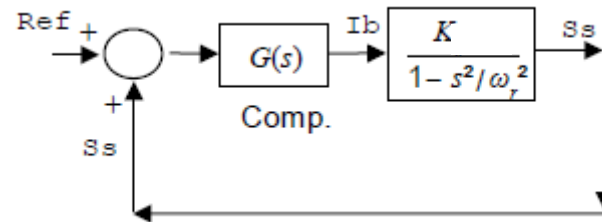


Diagrama de blocos simplificado

(K=Ganho de malha)

$$D(s) = \frac{s+z}{s+p} = \frac{s+21}{s+121}$$

$$G_{mf}(s) = \frac{-K\omega_r^2(s+z)}{s^3 + ps^2 + (K-1)\omega_r^2 s + (K-1)\omega_r^2}$$

## Ex. 2 idss

```
ze{i}=iddata(y-mean(y),u-mean(u),T);

gtf{i}=tfest(ze{i},3,1);          % Id. Caixa Preta – estrutura: 3 pólos 1 zero

% idss (fixa pólo em comp. Avnaço em -121) - 4 par. livres.
par=[-2e5;-7500;-1.3e4;-220]; % valores iniciais ~gtf

A = [0 1 0; 0 0 1; par(1) par(2) -121];
B = [0; 0; 1];
C = [par(3) par(4) 0];
D = zeros(1,1);
gs=idss(A,B,C,D);

gs.Structure.a.Free=[0 0 0; 0 0 0;1 1 0];
gs.Structure.b.Free=[0; 0; 0];
gs.Structure.c.Free=[1 1 0];
gs.Structure.d.Free=0;
set(gs,'Ts',0); % modelo contínuo
%gss = ssest(ze,gss,ssestOptions('Display','on'));
gss{i}=ssest(ze{i},gs);
```

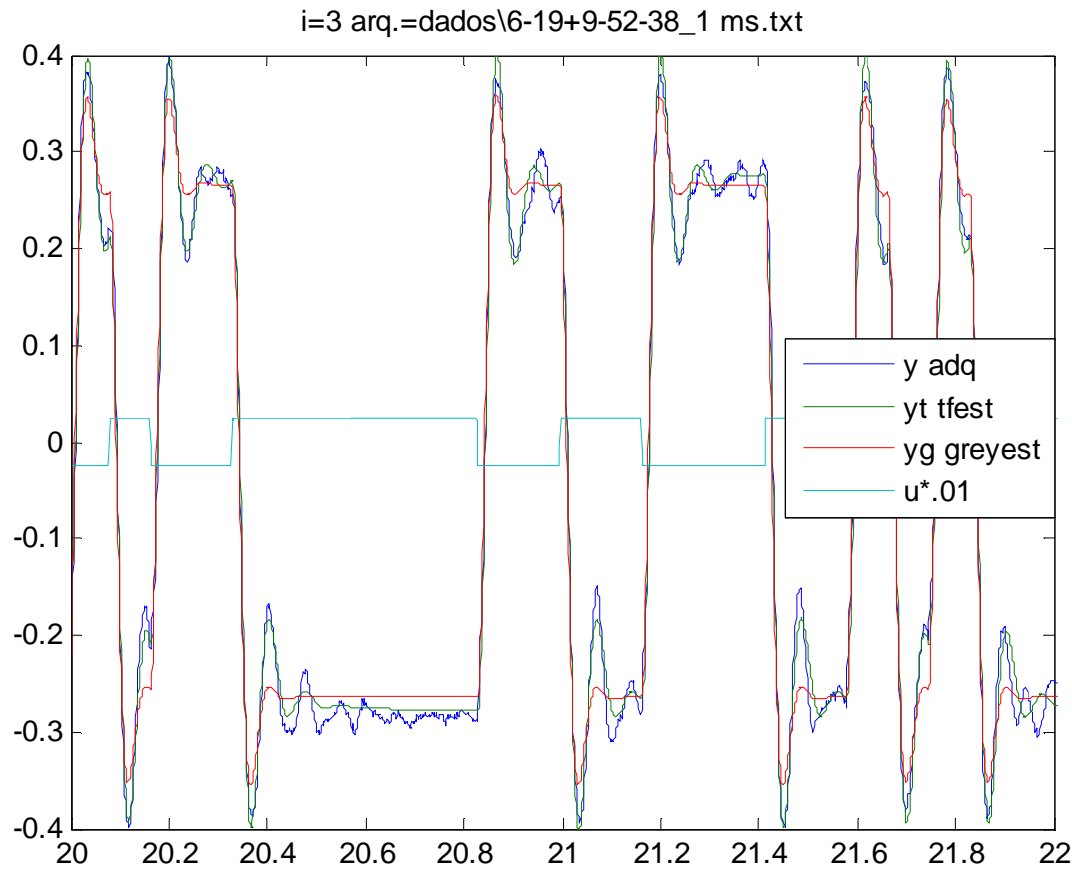
## Ex. 2 idgrey

```
par=[-2e5;-7500;-1.3e4];  
aux=21; % posição do zero - fixa!  
Ts=0; % modelo contínuo  
gr=idgrey('myLevi',par,'c',aux,Ts)  
  
ggr{i}=greyest(ze{i},gr)
```

```
function [A,B,C,D,K,x0] = myLevi(par,Ts,aux)  
A = [0 1 0; 0 0 1; par(1) par(2) -121];  
B = [0; 0; 1];  
C = [par(3)*aux par(3) 0];  
D = zeros(1,1);  
K = zeros(3,1);  
x0 =[0; 0; 0];
```

```
% Forma Canonica Controlável  
% b1 s + b0  
% -----  
% s^3 + a2 s^2 + a1 s + a0  
%  
%A=[0 1 0;  
% 0 0 1;  
% -a0 -a1 -a2]  
%B=[0;0;1];  
%C=[b0 b1 0]
```

# Ex. 2 Levitação Magnética



$$tf(s) = \frac{-726.9(s+17.11)}{s^3 + 67.46s^2 + 8560s + 1.13e05}$$

$$gr(s) = \frac{-1019.6(s+21)}{s^3 + 121s^2 + 1.164e04s + 2.05e05}$$

# MQR – MQ com Restrições

$$\mathbf{y} = \Psi \hat{\boldsymbol{\theta}} + \boldsymbol{\xi}$$

*minimizar a Função de Custo  $\boldsymbol{\xi}^T \boldsymbol{\xi}$*

*$n_r$  restrições sobre os parâmetros*

$$\mathbf{c} = S\boldsymbol{\theta}$$

$\mathbf{c} \in \mathbb{R}^{n_r \times 1}$  e  $S \in \mathbb{R}^{n_r \times n_\theta}$  são conhecidos.

$$\hat{\boldsymbol{\theta}}_{\text{MQR}} = \arg \min_{\boldsymbol{\theta} : \mathbf{c} = S\boldsymbol{\theta}} [\boldsymbol{\xi}^T \boldsymbol{\xi}]$$

$$\hat{\boldsymbol{\theta}}_{\text{MQR}} = (\Psi^T \Psi)^{-1} \Psi^T \mathbf{y} - (\Psi^T \Psi)^{-1} S^T [S (\Psi^T \Psi)^{-1} S^T]^{-1} (S \hat{\boldsymbol{\theta}}_{\text{MQ}} - \mathbf{c})$$

## Ex. MQR – Característica Estática

$$y(k) = 1,2796y(k-1) + 0,0178u(k-2)u(k-1) + 0,0408u(k-1)^2 - 0,3668y(k-2) - 0,2565u(k-2)y(k-1) + 0,2205u(k-2)y(k-2) + 0,0029u(k-2)^2, \quad ($$

$$\bar{y} = \frac{\Sigma_u^2 \bar{u}^2}{1 - \Sigma_y - \Sigma_{yu} \bar{u}}$$

$$\mathbf{c} = [\Sigma_u^2 \quad \Sigma_{yu} \quad \Sigma_y]^T = [0,0615 \quad -0,0360 \quad 0,9128]^T$$

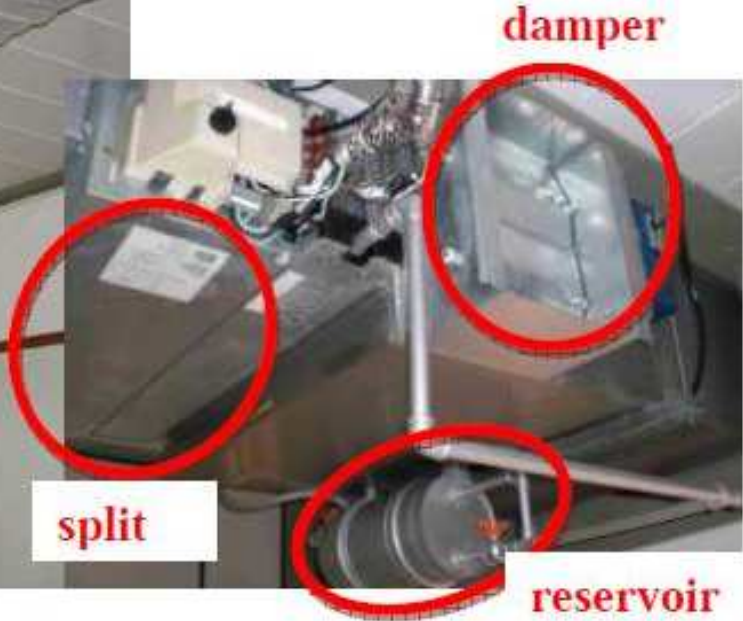
$$\begin{bmatrix} 0,0615 \\ -0,0360 \\ 0,9128 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \theta_7 \end{bmatrix}$$

$$\bar{y} = \frac{0,0615 \bar{u}^2}{1 - 0,9128 + 0,0360 \bar{u}}, \quad \text{Característica Estática}$$



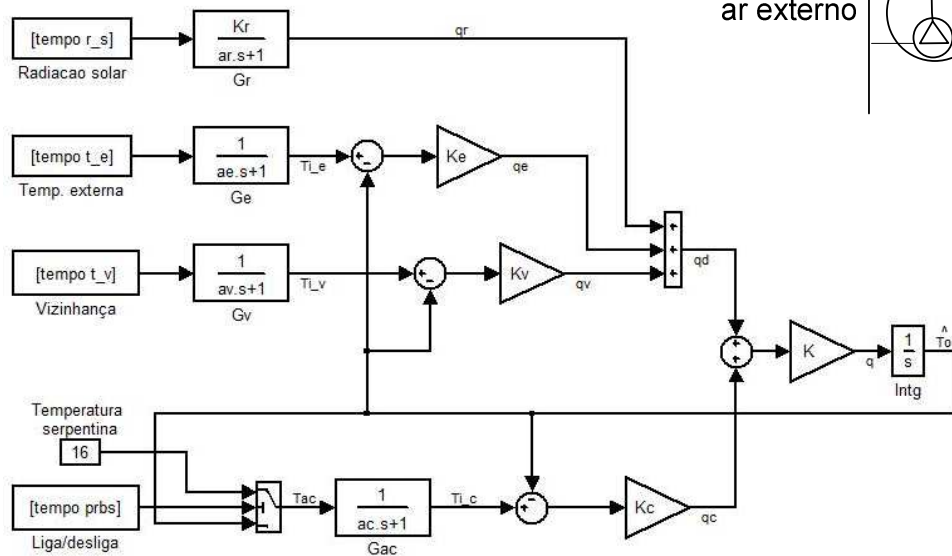
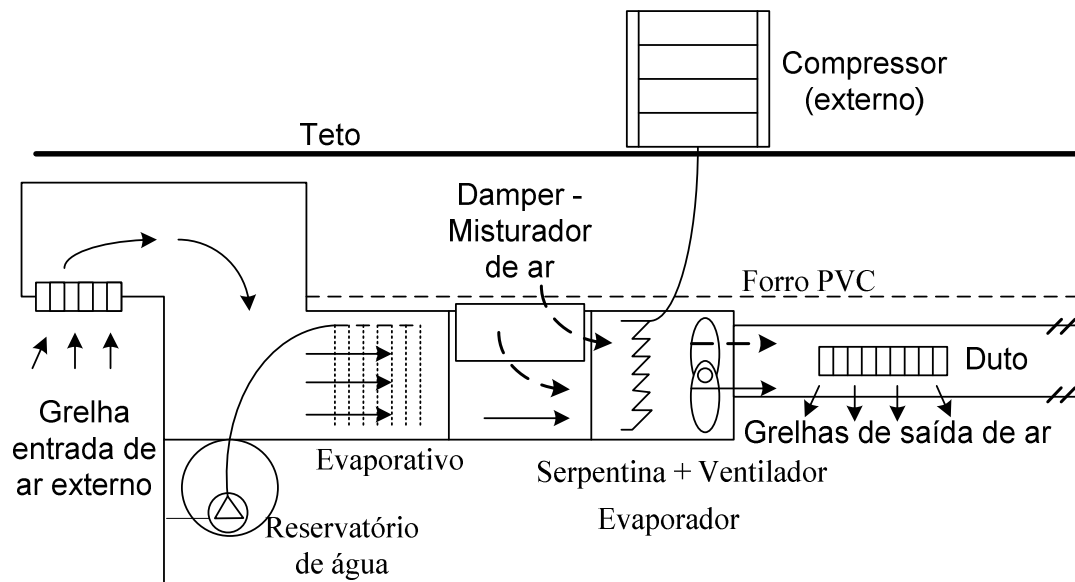
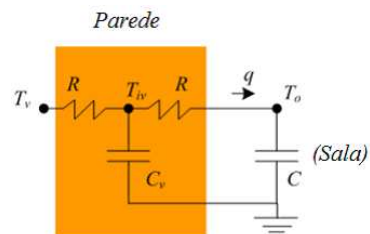
# Ex. Predial



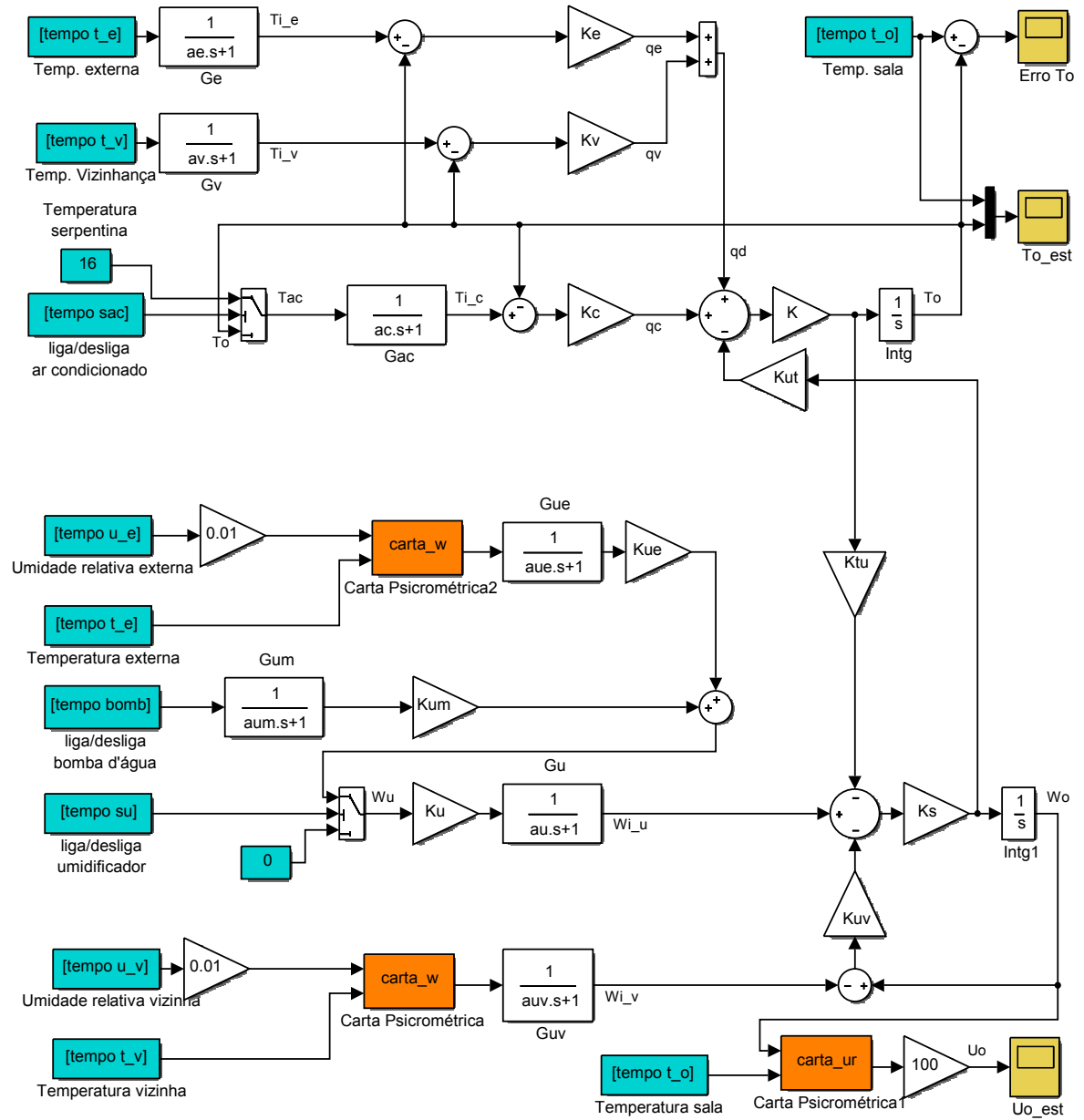
damper

split

reservoir



# T & UR



# idgrey

```
function [A,B,C,D,K,x0] = myfunc5(par,Ts,aux)
```

```
A = [par(1) par(2) par(3) par(4) par(5) par(6) par(7); 0 par(8) 0 0  
      0 0 0; 0 0 par(9) 0 0 0 0; 0 0 0 par(10) 0 0 0; 0 0 0 0 par(11)  
      par(12) par(13); 0 0 0 0 0 par(14) 0; 0 0 0 0 0 0 par(15)];
```

```
B = [par(16) par(17) par(18) par(19) par(20); (1-par(8)) 0 0 0 0; 0  
      (1-par(9)) 0 0 0; 0 0 (1-par(10)) 0 0; 0 0 0 par(21) par(22); 0 0  
      0 par(23) 0; 0 0 0 0 (1-par(15))];
```

```
C = [1 0 0 0 0 0 0; 0 0 0 0 1 0 0];
```

```
D = zeros(2,5);
```

```
K = zeros(7,2);
```

```
x0 =[par(24);par(25);par(26);par(27);par(28);par(29);par(30)];
```

$$A = \begin{bmatrix} a_{11} & \frac{K_e K}{\alpha} & \frac{K_v K}{\alpha} & \frac{K_c K}{\alpha} & a_{15} & \frac{-KK_{ut}K_s}{\alpha} & -a_{15} \\ 0 & -1/a_e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/a_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/a_c & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & K_s/\alpha & -a_{55} \\ 0 & 0 & 0 & 0 & 0 & -1/a_u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1/a_{uv} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/a_e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/a_v & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/a_c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_u/a_u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/a_{uv} & 0 \end{bmatrix}$$

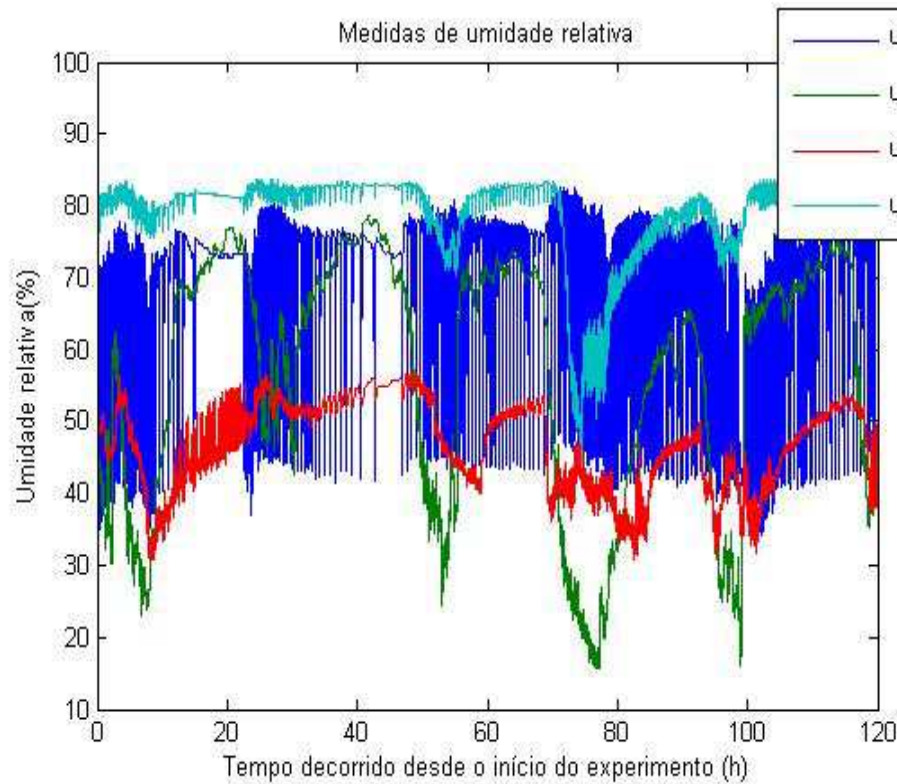
$$a_{11} = \frac{-K(K_e + K_v + K_c)}{\alpha}; a_{15} = \frac{KK_{ut}K_sK_{uv}}{\alpha};$$

$$a_{51} = \frac{KK_sK_{tu}(K_e + K_v + K_c)}{\alpha}; a_{52} = \frac{-K_eKK_sK_{tu}}{\alpha};$$

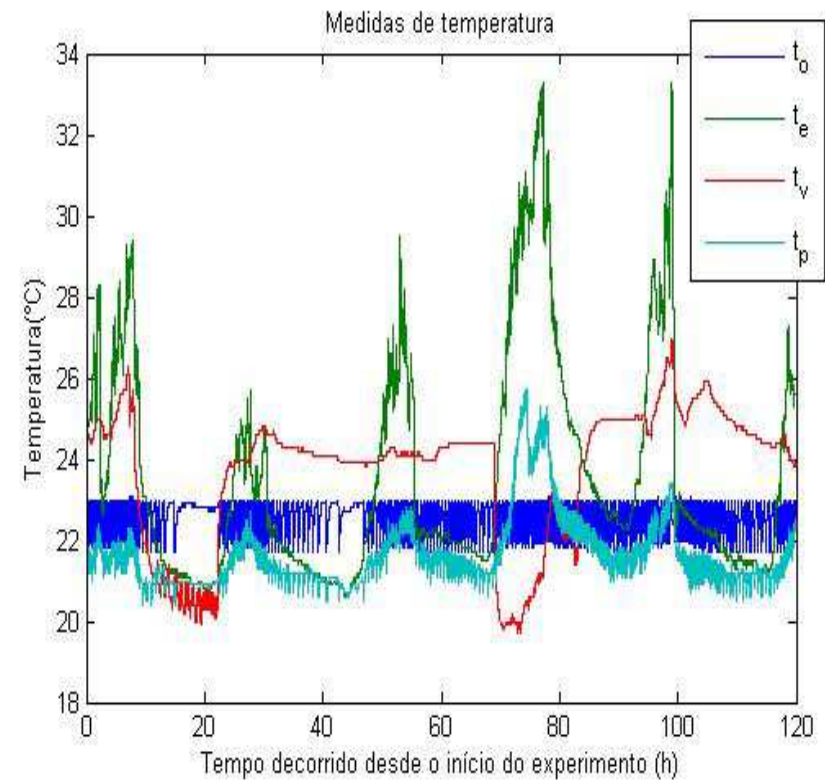
$$a_{53} = \frac{-K_vKK_sK_{tu}}{\alpha}; a_{54} = \frac{-K_cKK_sK_{tu}}{\alpha}; a_{55} = \frac{-K_sK_{uv}}{\alpha}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

# Predial - Sinais

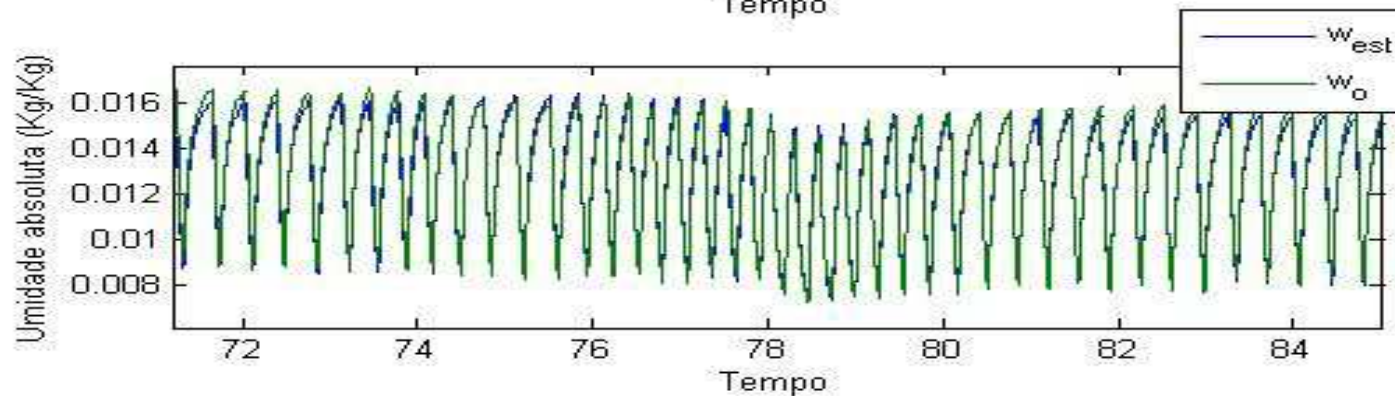
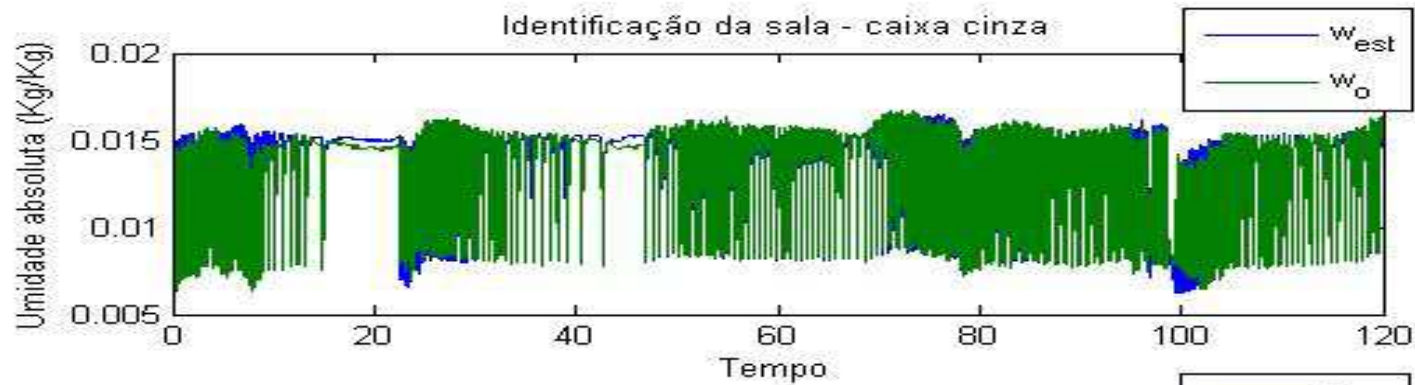


Umidade relativa da sala de reuniões ( $u_o$ ), do ambiente externo ( $u_e$ ), sala vizinha ( $u_v$ ) e pós-evaporativo ( $u_p$ ).



Temperatura da sala de reuniões ( $t_o$ ), do ambiente externo ( $t_e$ ), vizinhança ( $t_v$ ) e pós-evaporativo ( $t_p$ ).

# Resultado da Identificação



$i$	$u$	$uv$	$ue$	$um$	$s$
<b>ai</b>	168,5	164,9	80,82	171,7	-
<b>Ki</b>	2,19	3,77	0,151	-0,003	0,029

# Redução de Ordem

Utilização de Coeficientes de Agrupamento

- Modelo ARMAX

$$y(k) = \sum_{i=1}^{n_y} a_i(T_s) y[(k-i)T_s] + \sum_{i=1}^{n_u} b_i(T_s) u[(k-i)T_s] + \sum_{i=1}^{n_\xi} c_i(T_s) \xi[(k-i)T_s] + \xi(k).$$



# Candidatos a Modelo

$$y(t) = a_1^1(T_s)y(t - T_s) + b_1^1(T_s)u(t - T_s) + c_1^1(T_s)\tilde{\xi}(t - T_s) + \tilde{\xi}(t),$$

$$y(t) = a_1^2(T_s)y(t - T_s) + a_2^2(T_s)y(t - 2T_s) + b_1^2(T_s)u(t - T_s) \\ + b_2^2(T_s)u(t - 2T_s) + c_1^2(T_s)\xi(t - T_s) + \xi(t),$$

$$y(t) = a_1^3(T_s)y(t - T_s) + a_2^3(T_s)y(t - 2T_s) + a_3^3(T_s)y(t - 3T_s) \\ + b_1^3(T_s)u(t - T_s) + b_2^3(T_s)u(t - 2T_s) + b_3^3(T_s)u(t - 3T_s) \\ + c_1^3(T_s)\xi(t - T_s) + \xi(t),$$

Considerando:

pequenas períodos de amostragem

$$y(t - T_s) \approx y(t - 2T_s) \approx y(t - 3T_s)$$

$$y(t) = a_1^1(T_s)y(t - T_s) + b_1^1(T_s)u(t - T_s) + c_1^1(T_s)\tilde{\xi}(t - T_s) + \tilde{\xi}(k)$$

$$y(t) \approx (a_1^2(T_s) + a_2^2(T_s))y(t - T_s) + (b_1^2(T_s) + b_2^2(T_s))u(t - T_s) + c_1^2(T_s)\xi(t - T_s) + \xi(k)$$

$$y(t) \approx (a_1^3(T_s) + a_2^3(T_s) + a_3^3(T_s))y(t - T_s) + (b_1^3(T_s) + b_2^3(T_s) + b_3^3(T_s))u(t - T_s) + c_1^3(T_s)\xi(t - T_s) + \xi(k).$$

# Estimação dos Parâmetros

$$\begin{bmatrix} y(2) \\ y(3) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} y(1) & u(1) & \tilde{\xi}(1) \\ y(2) & u(2) & \tilde{\xi}(2) \\ \vdots & \vdots & \vdots \\ y(N-1) & u(N-1) & \tilde{\xi}(N-1) \end{bmatrix} \begin{bmatrix} \Sigma_y^{1,1}(T_s) \\ \Sigma_u^{1,1}(T_s) \\ c_1^1(T_s) \end{bmatrix} + \begin{bmatrix} \tilde{\xi}(2) \\ \tilde{\xi}(3) \\ \vdots \\ \tilde{\xi}(N) \end{bmatrix}$$

$$\begin{bmatrix} y(2) \\ y(3) \\ \vdots \\ y(N) \end{bmatrix} \approx \begin{bmatrix} y(1) & u(1) & \xi(1) \\ y(2) & u(2) & \xi(2) \\ \vdots & \vdots & \vdots \\ y(N-1) & u(N-1) & \xi(N-1) \end{bmatrix} \begin{bmatrix} \Sigma_y^{2,2}(T_s) \\ \Sigma_u^{2,2}(T_s) \\ c_1^2(T_s) \end{bmatrix} + \begin{bmatrix} \xi(2) \\ \xi(3) \\ \vdots \\ \xi(N) \end{bmatrix}$$

$$\begin{bmatrix} y(2) \\ y(3) \\ \vdots \\ y(N) \end{bmatrix} \approx \begin{bmatrix} y(1) & u(1) & \xi(1) \\ y(2) & u(2) & \xi(2) \\ \vdots & \vdots & \vdots \\ y(N-1) & u(N-1) & \xi(N-1) \end{bmatrix} \begin{bmatrix} \Sigma_y^{3,3}(T_s) \\ \Sigma_u^{3,3}(T_s) \\ c_1^3(T_s) \end{bmatrix} + \begin{bmatrix} \xi(2) \\ \xi(3) \\ \vdots \\ \xi(N) \end{bmatrix}$$

$$\Sigma_y^{1,1}(T_s) \neq \Sigma_y^{2,2}(T_s) \approx \Sigma_y^{3,3}(T_s) \approx \dots \approx \Sigma_y^{q,q}(T_s)$$

# Redução de Ordem

- Encontrar o “joelho”  
acrécimo de termo é irrelevante:

$$| \Sigma_y(T_s) - \Sigma_{yo}(T_s) | < \gamma$$