

Identificação de Sistemas Dinâmicos

Filtro de Kalman

Adaptado de:

Aguirre, Cap. 9 - Introdução à Identificação de Sistemas,
Ed. UFMG, 3ª edição

Introdução

- Modelo Linear Discreto

$$\begin{cases} \mathbf{x}_{k+1} &= \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k \\ \mathbf{y}_{k+1} &= H_{k+1} \mathbf{x}_{k+1} + \mathbf{v}_{k+1}, \end{cases}$$

$$E[\mathbf{w}_k \mathbf{w}_k^T] = Q_k, \quad E[\mathbf{v}_k \mathbf{v}_k^T] = R_k$$

$$E[\mathbf{v}_i \mathbf{w}_j^T] = 0, \quad \forall i, j.$$

Condições Iniciais $\hat{\mathbf{x}}_0, P_0$

seqüência de entrada $\mathbf{u}_k \in \mathfrak{R}^r$ e *medições* $\mathbf{y}_k \in \mathfrak{R}^m$

Função de Probabilidade

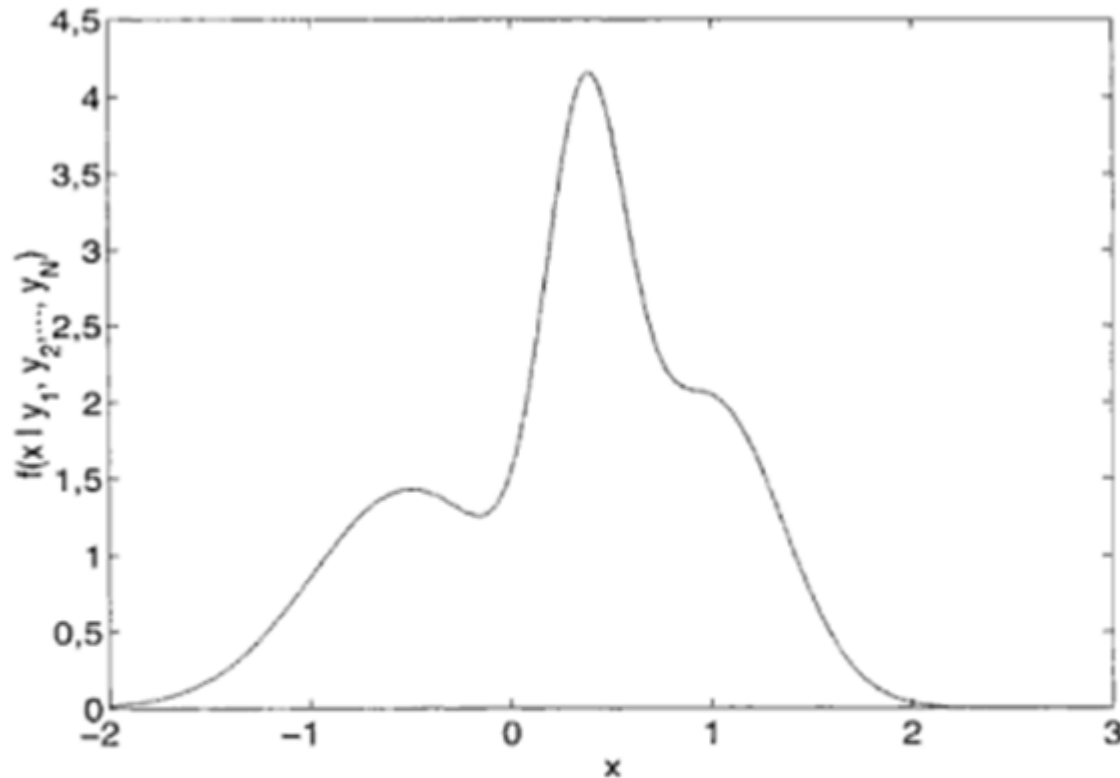
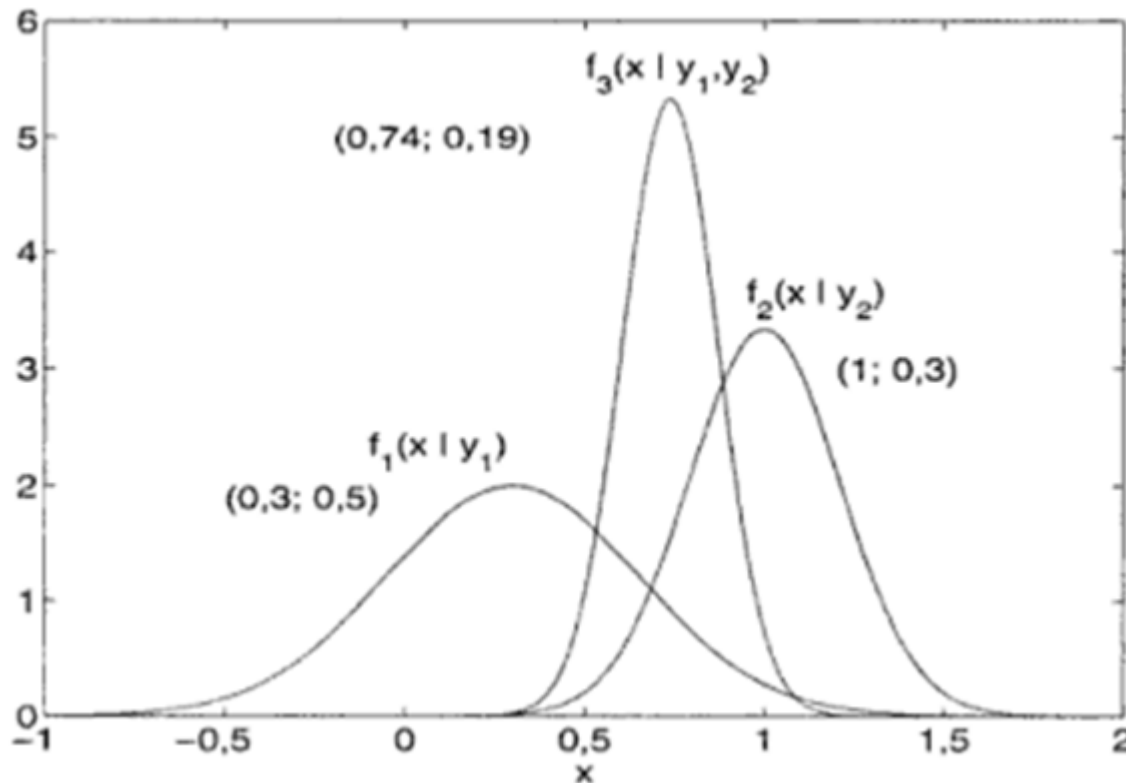


FIGURA 9.1: Função de probabilidade geral

Função de densidade de probabilidade condicional de x dadas as medições y_i , $i = 1, \dots, N$. No presente caso hipotético, nota-se que valores em torno de $x = 0,4$ são os mais prováveis. O fato de $f(x|y_1, \dots, y_N)$ ser bastante dispersa em torno de $x = 0,4$ indica pouca confiança nas medições.

Função de densidade de Probabilidade Gaussiana

$$f_{\mathbf{z}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |P|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \bar{\mathbf{z}})^T P^{-1} (\mathbf{x} - \bar{\mathbf{z}}) \right\} \quad f_z(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{1}{2\sqrt{\sigma}} (x - \bar{z})^2 \right\}$$



$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} y_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} y_2$$
$$\sigma^2 = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1}$$

Combinação de 2 sensores

$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} y_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} y_2$$

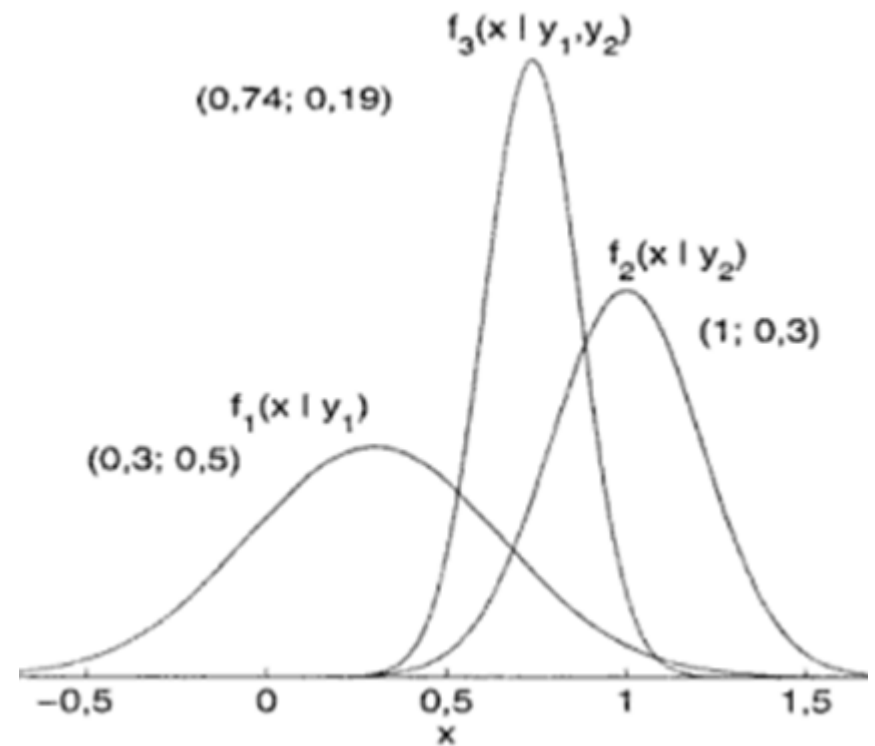
$$\sigma^2 = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1}$$

$$\hat{x}_2 = \frac{\sigma_2^2 y_1 + \sigma_1^2 y_2 + \sigma_1^2 y_1 - \sigma_1^2 y_1}{\sigma_1^2 + \sigma_2^2}$$

$$= \frac{(\sigma_1^2 + \sigma_2^2) y_1 + \sigma_1^2 (y_2 - y_1)}{\sigma_1^2 + \sigma_2^2}$$

$$= y_1 + K (y_2 - y_1),$$

$$K = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \quad \text{e} \quad \sigma_2^2 = \sigma_1^2 - K \sigma_1^2$$



Medidas de um mesmo instrumento

$$\hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[y(t_2) - \hat{x}(t_1)],$$

$$K(t_2) = \frac{\sigma^2(t_1)}{\sigma^2(t_1) + \sigma^2(t_2)}$$

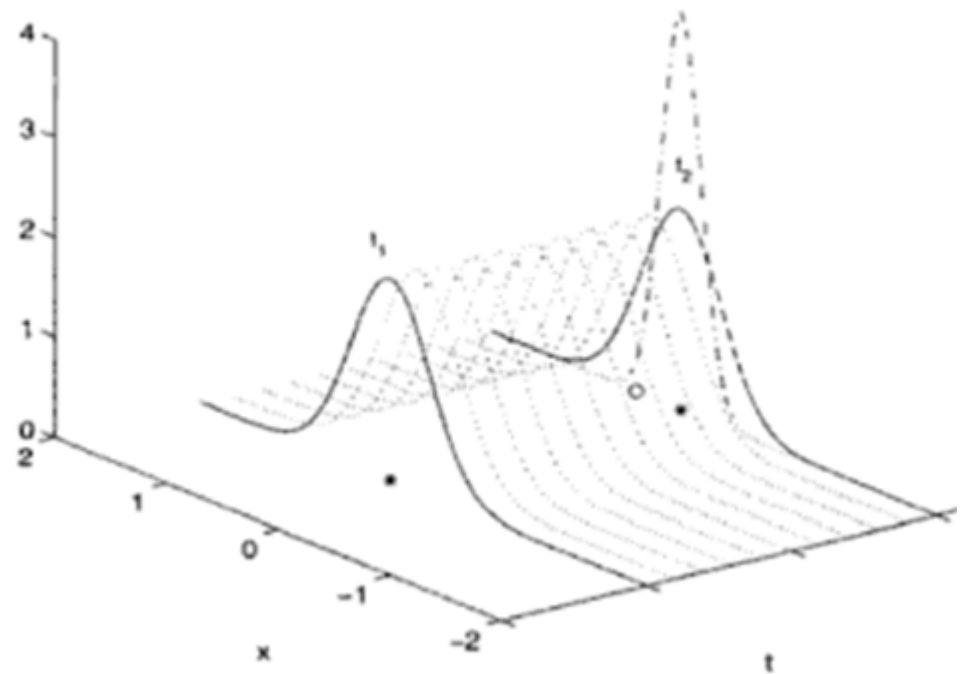
$$\sigma^2(t_2) = \sigma^2(t_1) - K(t_2)\sigma^2(t_1).$$

Nova Nomenclatura

$$\hat{x}(t_2|t_2) = \hat{x}(t_2|t_1) + K(t_2)[y(t_2) - \hat{x}(t_2|t_1)]$$

$$K(t_2) = \frac{\sigma^2(t_2|t_1)}{\sigma^2(t_2|t_1) + \sigma^2(t_2)},$$

$$\sigma^2(t_2|t_2) = \sigma^2(t_2|t_1) - K(t_2)\sigma^2(t_2|t_1).$$



O Caso Dinâmico

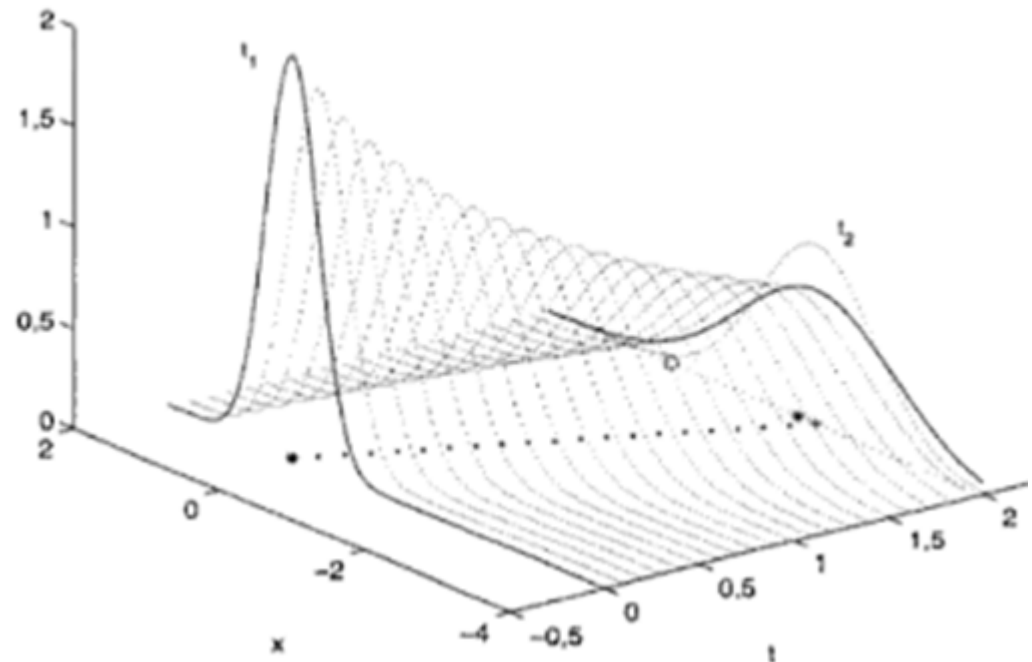
Movimento do Objeto:

e.g. Aproximação de Euler

$$\frac{x(t_2) - x(t_1)}{t_2 - t_1} \approx v + w(t_2)$$
$$x(t_2) \approx x(t_1) + (t_2 - t_1)[v + w(t_2)],$$

$$\hat{x}(t_2|t_1) = \hat{x}(t_1|t_1) + (t_2 - t_1)v$$

$$\sigma^2(t_2|t_1) = \sigma^2(t_1|t_1) + (t_2 - t_1)\sigma_w^2$$



O Filtro de Kalman Discreto

Lei de Movimento – propagação do vetor de estados estimado:

$$\hat{\mathbf{x}}(t_2|t_1) = \Phi(t_1)\hat{\mathbf{x}}(t_1|t_1) + \Gamma(t_1)\mathbf{u}(t_1).$$

Nova Nomenclatura

$$\hat{\mathbf{x}}_{k+1}^- = \Phi_k \hat{\mathbf{x}}_k^+ + \Gamma_k \mathbf{u}_k$$

Predição

$$\hat{\mathbf{x}}_{k+1}^+ = \hat{\mathbf{x}}_{k+1}^- + K_{k+1} [\mathbf{y}_{k+1} - H_{k+1} \hat{\mathbf{x}}_{k+1}^-]$$

Correção

Matriz de Covariância

- Erro de estimação na propagação $\hat{\mathbf{x}}_{k+1}^- = \hat{\mathbf{x}}_{k+1}^- - \mathbf{x}_{k+1}$

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1}^- &= [\Phi_k \hat{\mathbf{x}}_k^+ + \Gamma_k \mathbf{u}_k] - [\Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k] \\ &= \Phi_k (\hat{\mathbf{x}}_k^+ - \mathbf{x}_k) - \Upsilon_k \mathbf{w}_k \\ &= \Phi_k \tilde{\mathbf{x}}_k^+ - \Upsilon_k \mathbf{w}_k.\end{aligned}$$

Matriz de covariância de $\hat{\mathbf{x}}_{k+1}^-$

$$\begin{aligned}P_{k+1}^- &= E[(\hat{\mathbf{x}}_{k+1}^- - E[\hat{\mathbf{x}}_{k+1}^-])(\hat{\mathbf{x}}_{k+1}^- - E[\hat{\mathbf{x}}_{k+1}^-])^T] \\ &= E[(\hat{\mathbf{x}}_{k+1}^- - \mathbf{x}_{k+1})(\hat{\mathbf{x}}_{k+1}^- - \mathbf{x}_{k+1})^T] \\ &= E[\tilde{\mathbf{x}}_{k+1}^- (\tilde{\mathbf{x}}_{k+1}^-)^T],\end{aligned}$$

...

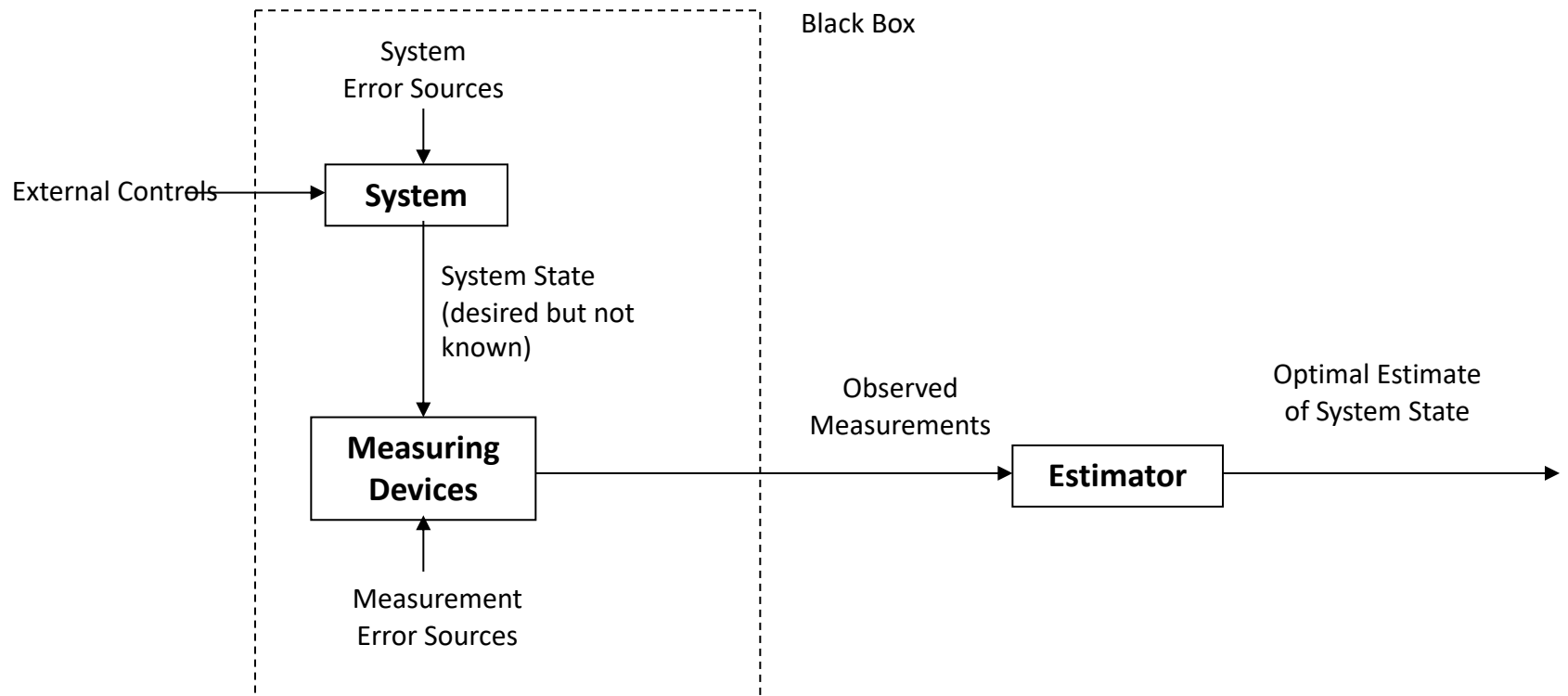
Equações de Filtro de Kalman Discreto

$$\left\{ \begin{array}{l}
 \hat{\mathbf{x}}_{k+1}^- = \Phi_k \hat{\mathbf{x}}_k^+ + \Gamma_k \mathbf{u}_k; \\
 P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T; \\
 K_{k+1} = P_{k+1}^- H_{k+1}^T [H_{k+1} P_{k+1}^- H_{k+1}^T + R_{k+1}]^{-1}; \\
 \hat{\mathbf{x}}_{k+1}^+ = \hat{\mathbf{x}}_{k+1}^- + K_{k+1} [\mathbf{y}_{k+1} - H_{k+1} \hat{\mathbf{x}}_{k+1}^-]; \\
 P_{k+1}^+ = P_{k+1}^- - K_{k+1} H_{k+1} P_{k+1}^-;
 \end{array} \right. \begin{array}{l} \text{Predição} \\ \\ \\ \text{Correção} \end{array}$$

$$\left\{ \begin{array}{l}
 \hat{\mathbf{x}}_{k+1} = \Phi_k \hat{\mathbf{x}}_k + \Gamma_k \mathbf{u}_k + \Phi_k K_k [\mathbf{y}_k - H_k \hat{\mathbf{x}}_k]; \\
 K_k = P_k H_k^T [H_k P_k H_k^T + R_k]^{-1}; \\
 P_{k+1} = \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T,
 \end{array} \right.$$

Acesso às variáveis de estado:

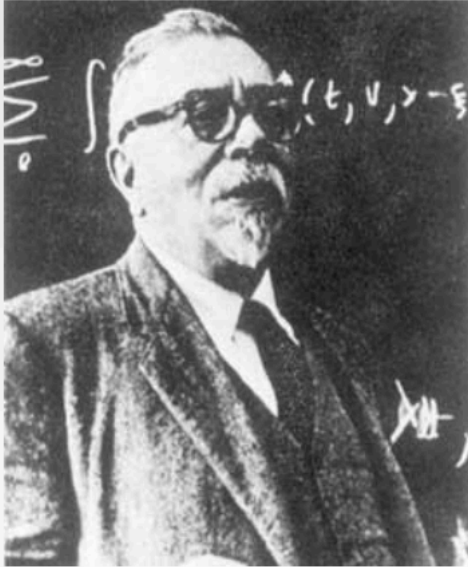
- As variáveis de estado não podem ser medidas diretamente
- Qual a melhor estimativa - “ótima” a partir das medidas?



Abordagens:

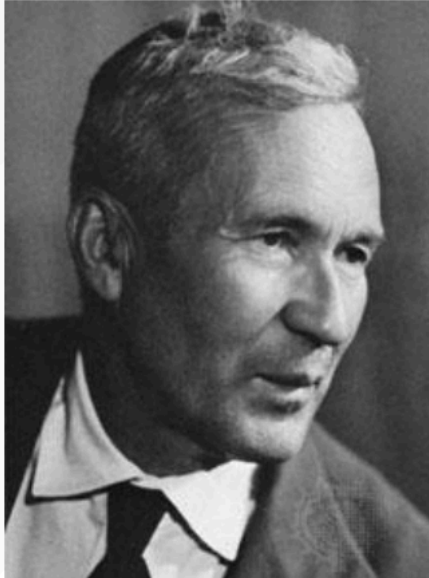
i) derivar y , ii) Modelo Malha aberta, iii) Observador, iv) Filtro de Kalman

A História da Filtragem de Sinais



1930

**Filtro de Wiener
(contínuo)**



1940

**Filtro de Kolmogorov
(discreto)**



1960

**Filtro de Kalman
(linear e estendido)**

Gaussiana

Para duas variáveis – Distribuição Espacial

$$f(x, y) = \frac{e^{-\frac{[(x-\bar{x}) \ (y-\bar{y})](\sigma_{xy}^2)^{-1} \begin{bmatrix} (x-\bar{x}) \\ (y-\bar{y}) \end{bmatrix}}{2}}}{\sqrt{2\pi\sigma_{xy}^2}}$$

Média de **x**: \bar{x}

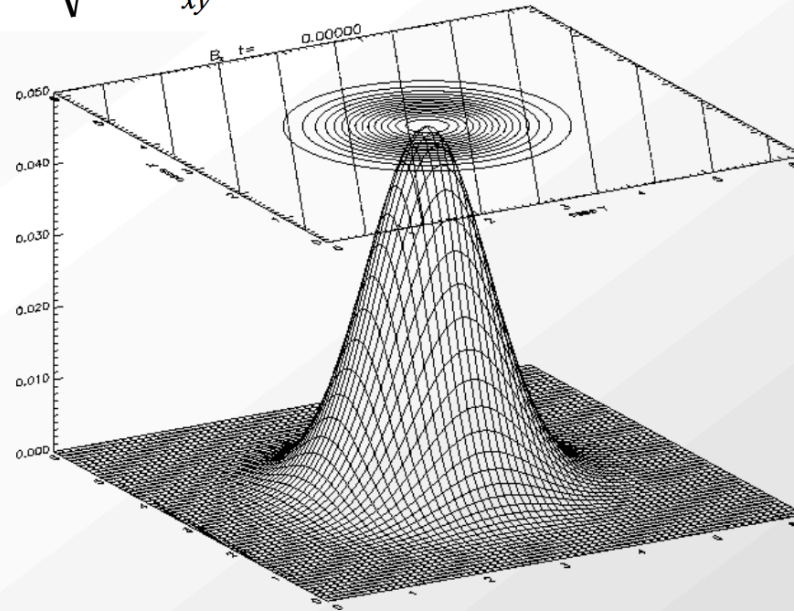
Média de **y**: \bar{y}

Desvio Padrão de **x**: σ_x^2

Desvio padrão de **y**: σ_y^2

Covariância de **x** e **y**: σ_{xy}^2

Correlação de **x** e **y**: ρ



Mas a Covariância é uma matriz (P)

- É necessário uma correção na fórmula pois,

$$P = \sigma_{xy}^2 = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}$$

A função gaussina corrigida para duas variáveis

$$f(x, y) = \frac{e^{-\frac{[(x-\bar{x}) \quad (y-\bar{y})] P^{-1} \begin{bmatrix} (x-\bar{x}) \\ (y-\bar{y}) \end{bmatrix}}{2}}}{\sqrt{2\pi} |P|^{1/2}}$$

onde $|P|$ é o determinante da matriz de covariância

Problema

- E se as variáveis forem vetores de dados com sinais do tipo:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- A covariância seria:

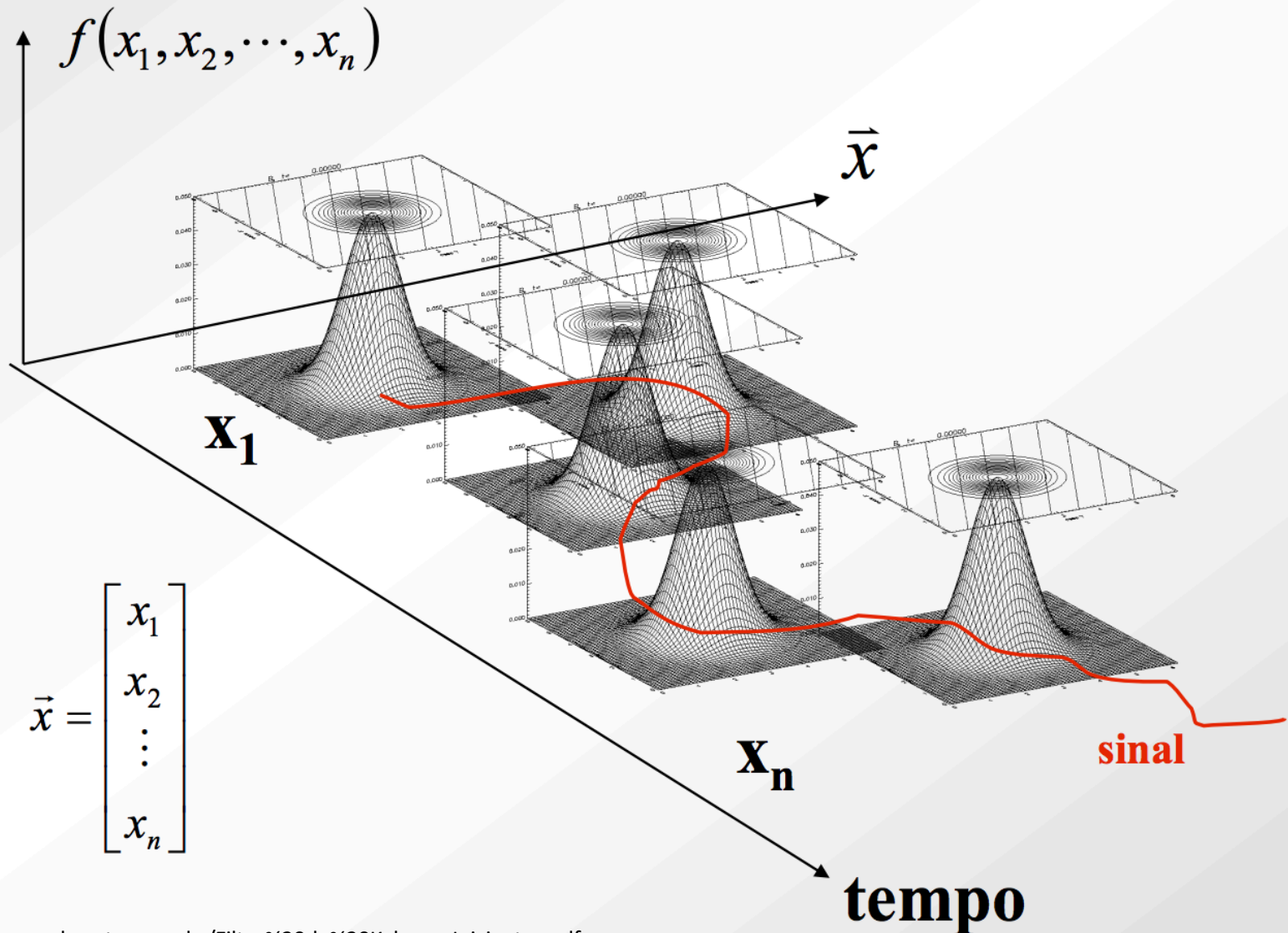
$$P = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \dots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & & \\ \vdots & & \ddots & \\ & & & \sigma_n^2 \end{pmatrix}$$

Impraticável! Todos os sinais teriam que serem armazenados em gigantescos bancos de dados para o calculo no fim.

Solução: Filtro de Kalman



Como o filtro de Kalman estima variáveis?



Algoritmo Esquemático

- Trabalha com dados
- Equação de Propagação
- Equação de Atualização
- Equação de Covariância

Satélite



dados



Bóia

Equação de Atualização \hat{x}_{k+1}^+

Equação de Atualização \hat{x}_{k+2}^+

k

k+1

k+2

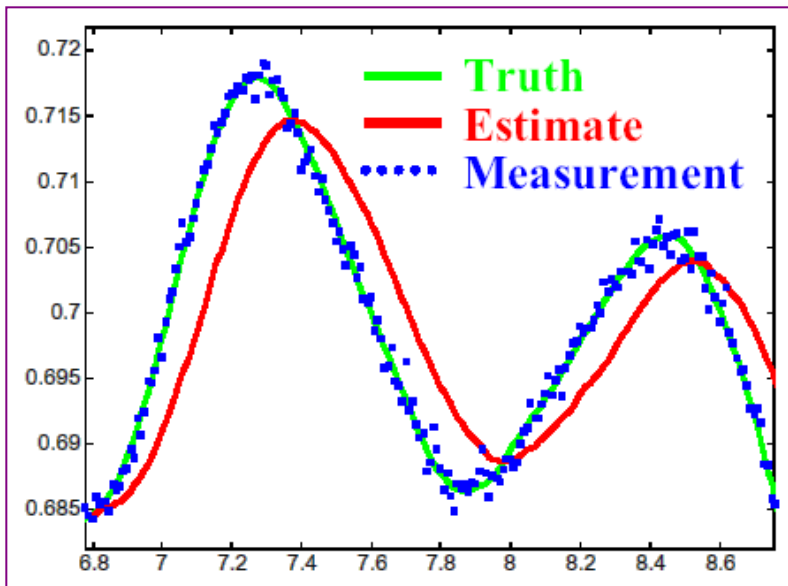
tempo

Equação de Propagação \hat{x}_{k+1}^-

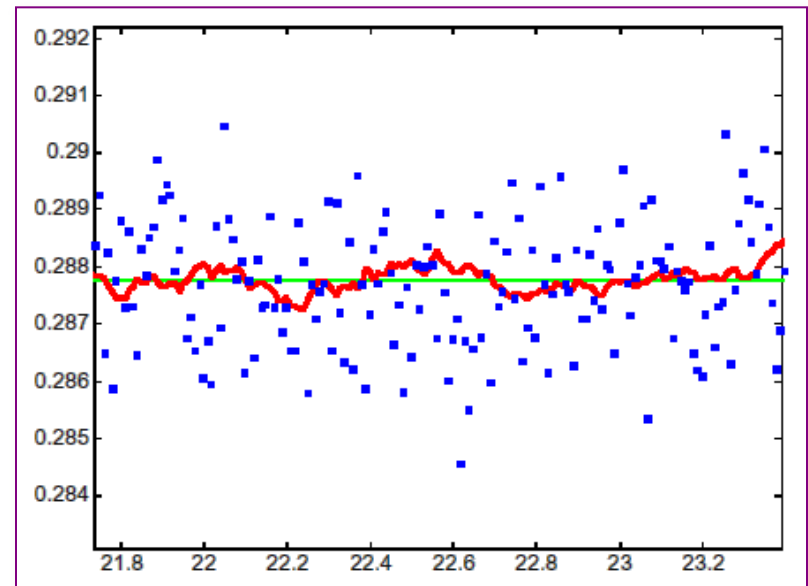
Equação de Propagação \hat{x}_{k+2}^-

Equação de Propagação \hat{x}_{k+3}^-

Exemplo: Apenas modelo de Posição

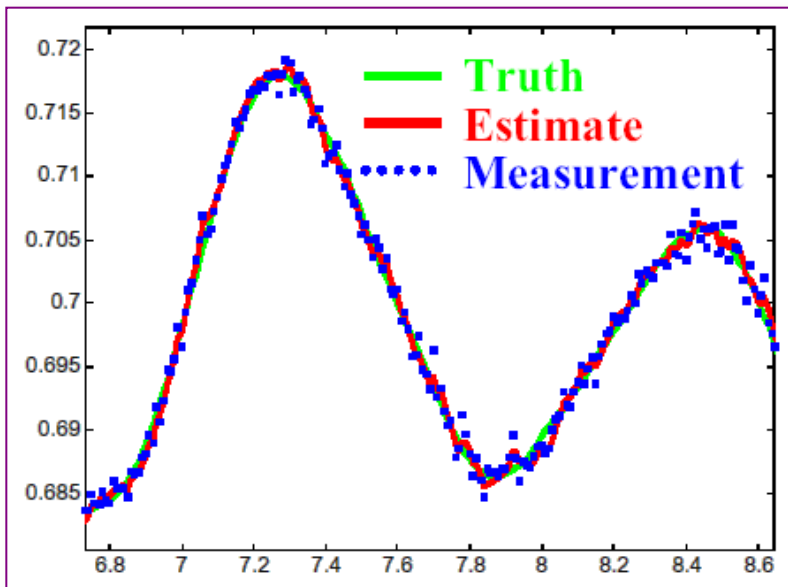


Moving

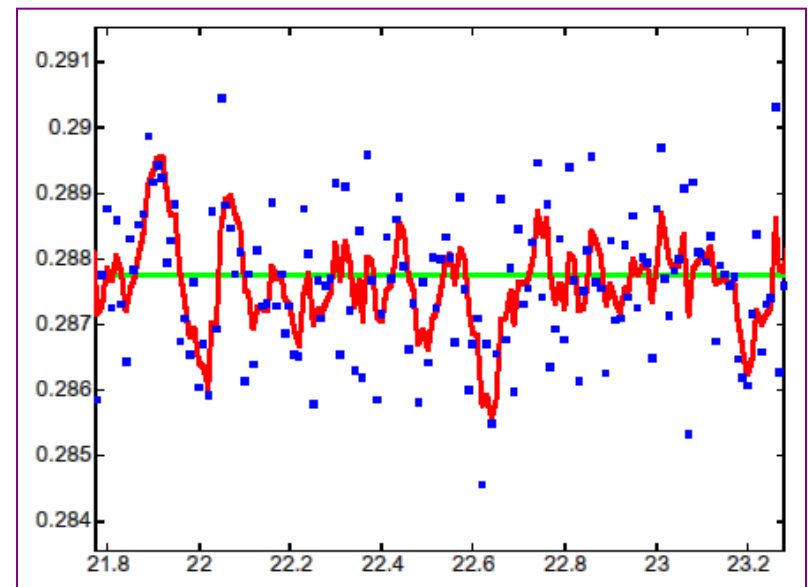


Still

Exemplo: Modelo de Posição e Velocidade

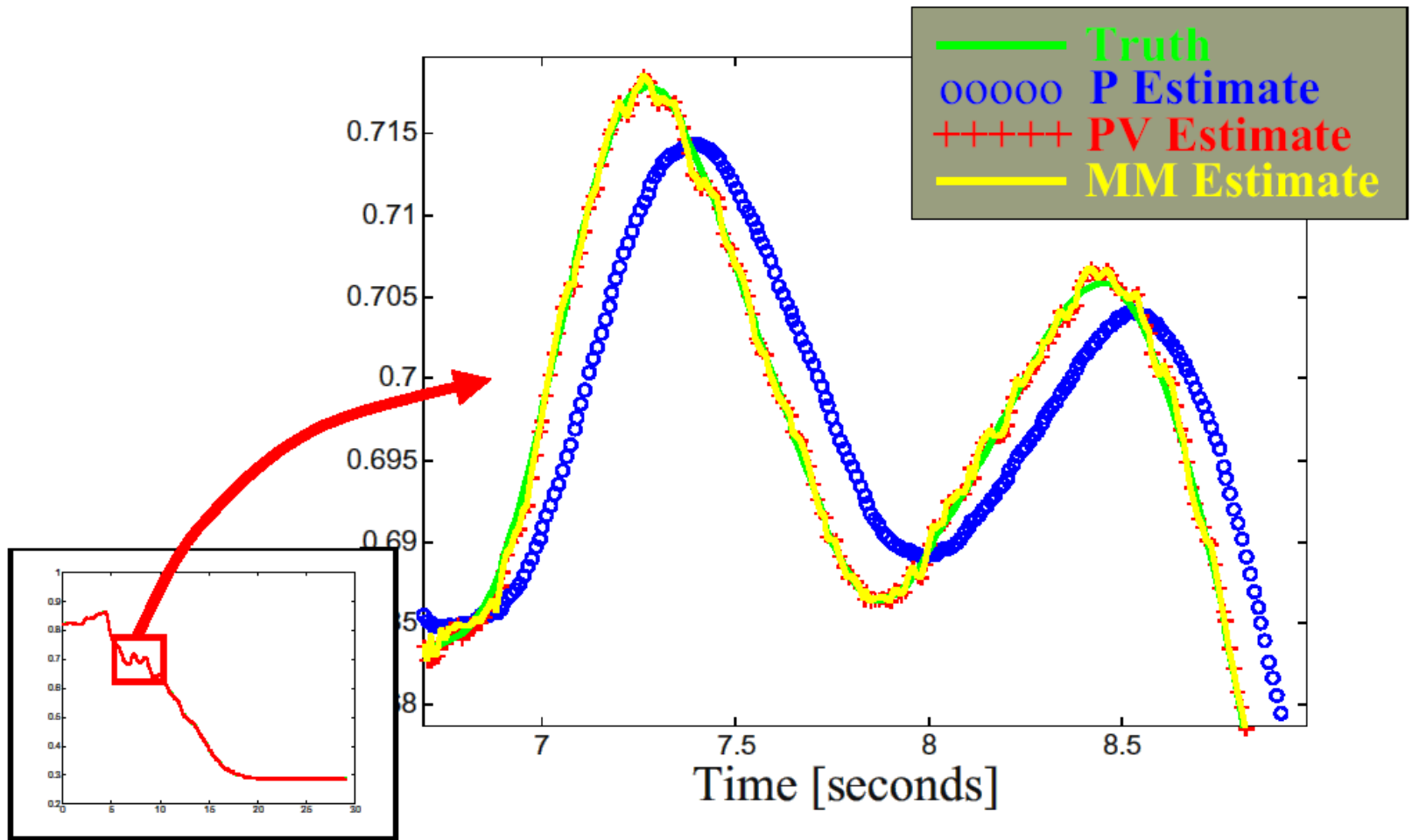


Moving



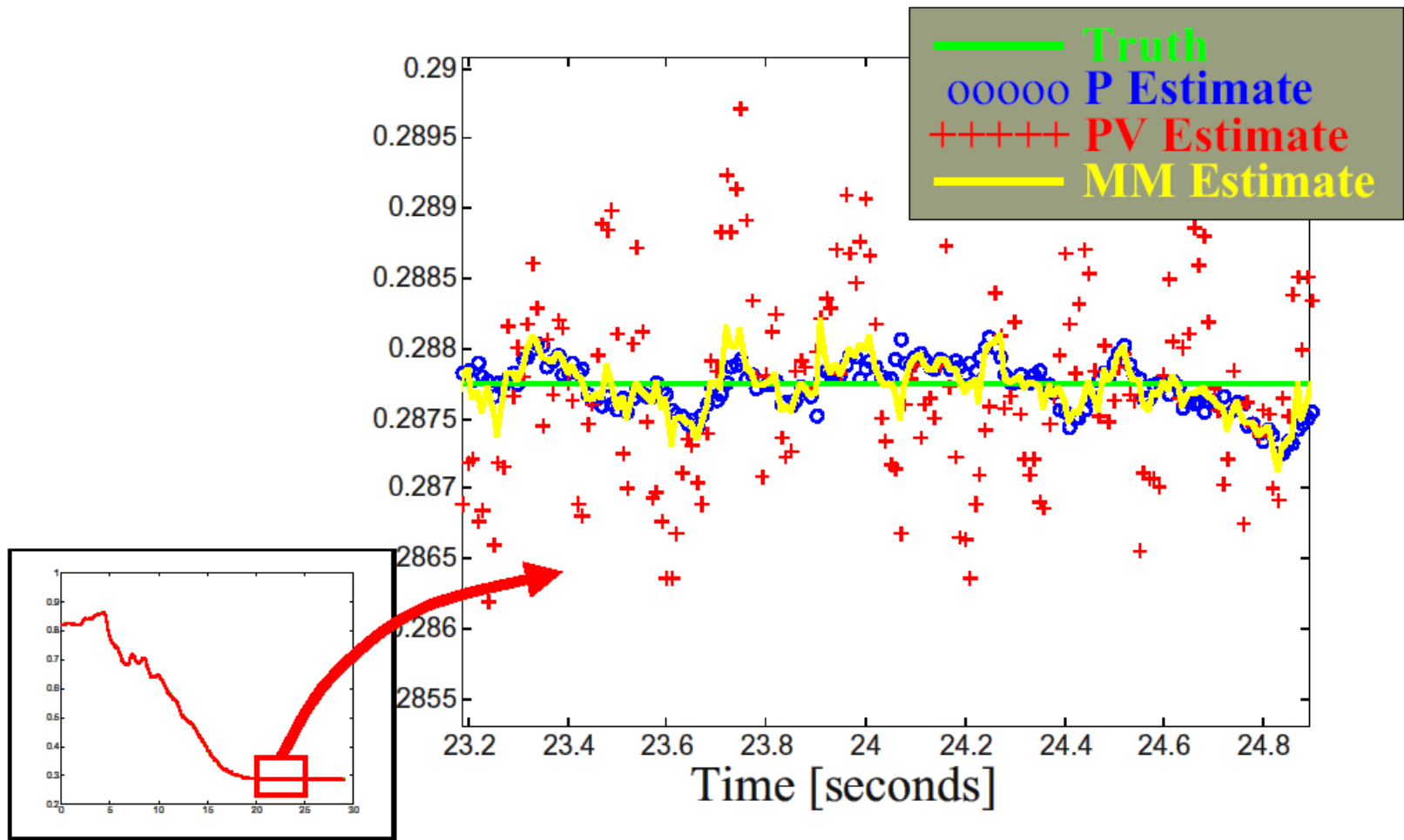
Still

Filtro Multi-modelo

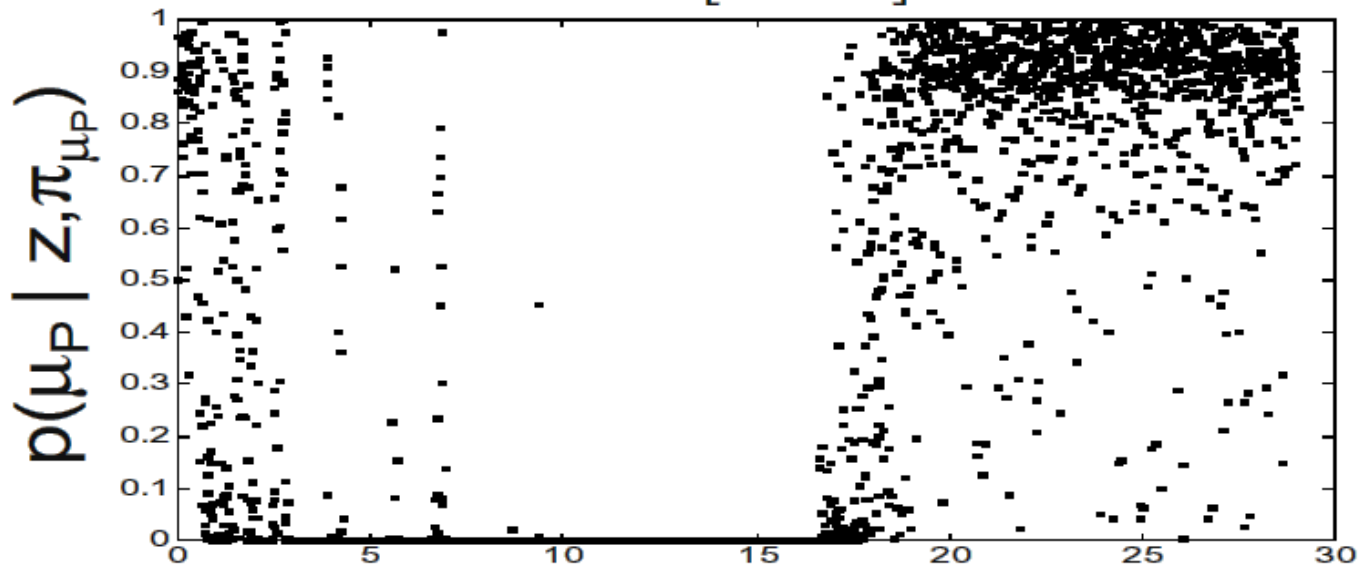
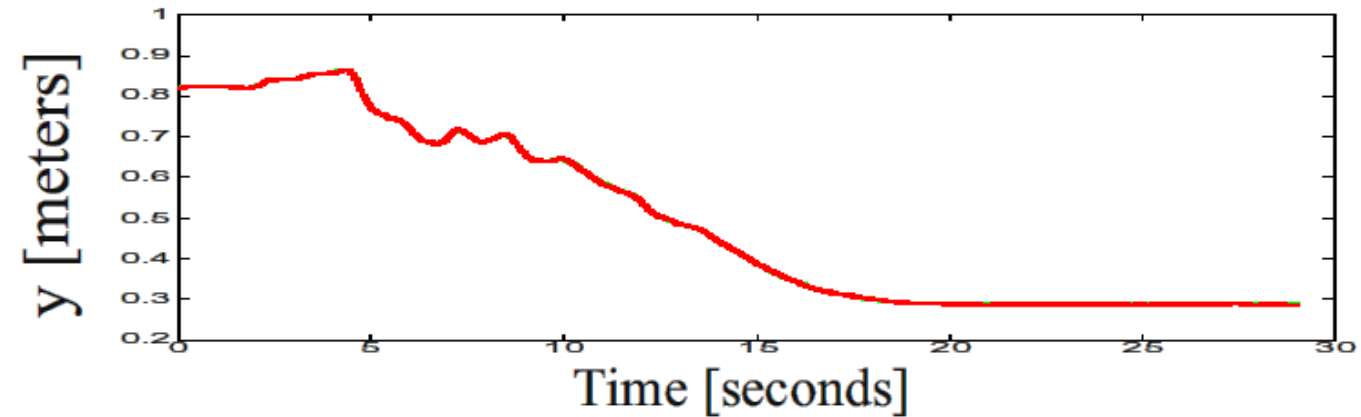


[Welch & Bishop]

Filtro Multi-modelo

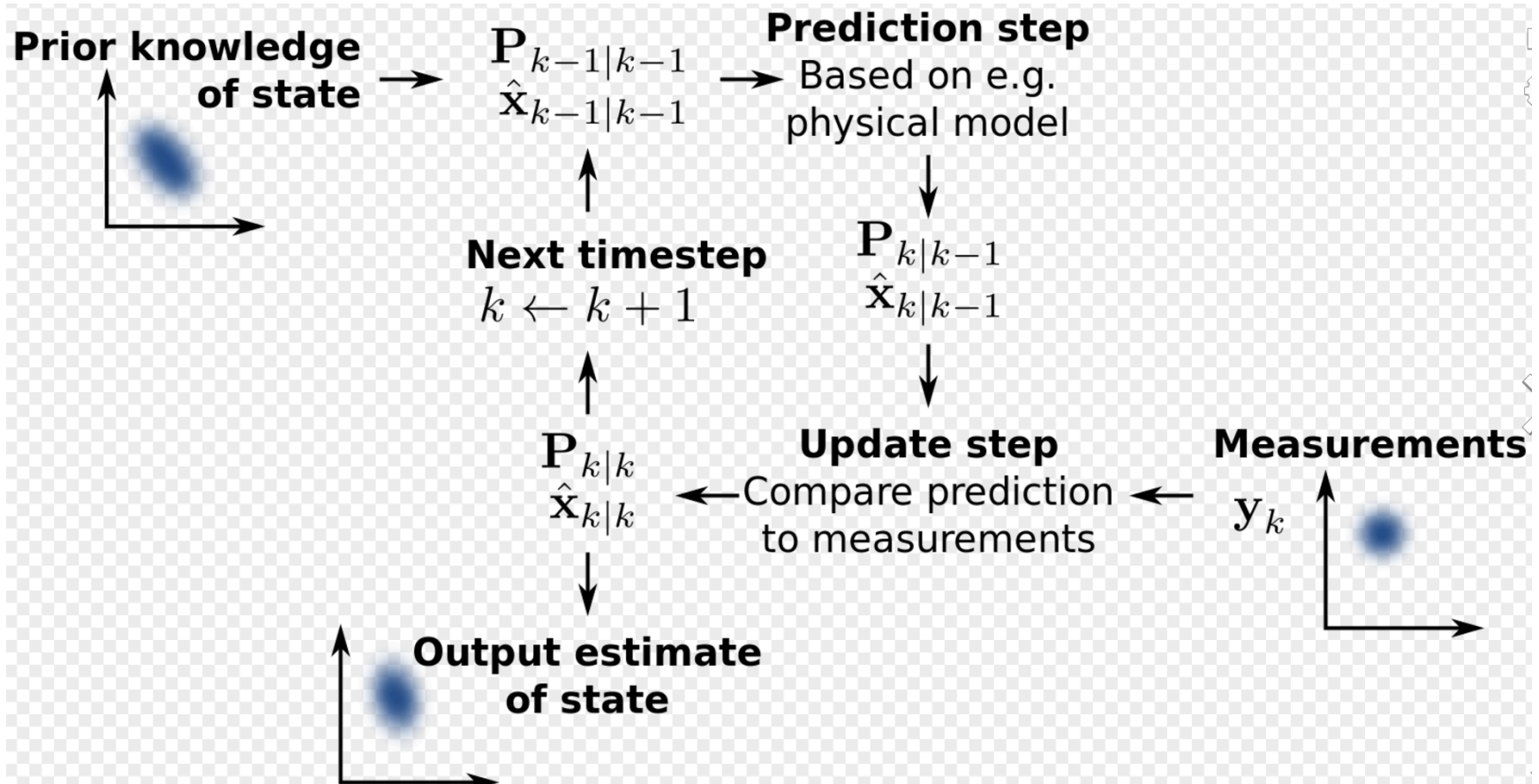


Results: Multiple Models



[Welch & Bishop]

Filtro de Kalman



Equações de Filtro de Kalman Discreto

$$\left\{ \begin{array}{l}
 \hat{\mathbf{x}}_{k+1}^- = \Phi_k \hat{\mathbf{x}}_k^+ + \Gamma_k \mathbf{u}_k; \\
 P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T; \\
 K_{k+1} = P_{k+1}^- H_{k+1}^T [H_{k+1} P_{k+1}^- H_{k+1}^T + R_{k+1}]^{-1}; \\
 \hat{\mathbf{x}}_{k+1}^+ = \hat{\mathbf{x}}_{k+1}^- + K_{k+1} [\mathbf{y}_{k+1} - H_{k+1} \hat{\mathbf{x}}_{k+1}^-]; \\
 P_{k+1}^+ = P_{k+1}^- - K_{k+1} H_{k+1} P_{k+1}^-;
 \end{array} \right. \begin{array}{l} \text{Predição} \\ \\ \\ \text{Correção} \end{array}$$

$$\left\{ \begin{array}{l}
 \hat{\mathbf{x}}_{k+1} = \Phi_k \hat{\mathbf{x}}_k + \Gamma_k \mathbf{u}_k + \Phi_k K_k [\mathbf{y}_k - H_k \hat{\mathbf{x}}_k]; \\
 K_k = P_k H_k^T [H_k P_k H_k^T + R_k]^{-1}; \\
 P_{k+1} = \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T,
 \end{array} \right.$$

Estimação de Parâmetros - Filtro de Kalman

Modelo no Espaço de Estados

$$\begin{cases} x_k = \Phi x_{k-1} + \Gamma u_k + w_k \\ y_k = Hx_{k-1} + v_k \end{cases}$$

Sinais

Filtro de Kalman

$$\begin{cases} K_k = \Phi P_{k-1} \Phi' H \psi' [H \Phi P_{k-1} \Phi' H' + R_k]^{-1} \\ \hat{x}_k = \Phi \hat{x}_{k-1} + \Gamma u_{k-1} + K_k [y_k - H(\Phi \hat{x}_{k-1} + \Gamma u_{k-1})] \\ P_k = \Phi P_{k-1} \Phi' - K_k H \Phi P_{k-1} \Phi' \end{cases}$$

$$\begin{cases} \theta_k = \theta_{k-1} \\ y_k = \psi_k^T \theta_{k-1} + e_k \end{cases}$$

Parâmetros

$$\begin{cases} K_k = P_{k-1} \psi_k [\psi_k' P_{k-1} \psi_k + R_k]^{-1} \\ \hat{\theta}_k = \hat{\theta}_{k-1} + K_k [y_k - \psi_k' \hat{\theta}_{k-1}] \\ P_k = P_{k-1} - K_k \psi_k' P_{k-1} \end{cases}$$

Variante do Filtro de Kalman

$$a) \begin{cases} \mathbf{x}_{k+1} = \Phi \mathbf{x}_k + \Gamma \mathbf{u}_k + \mathbf{w}_k \\ \mathbf{y}_k = H \mathbf{x}_k + \mathbf{v}_k \end{cases}$$

Variáveis

$$\begin{cases} \hat{\mathbf{x}}_{k+1} = \Phi_k \hat{\mathbf{x}}_k + \Gamma_k \mathbf{u}_k + \Phi_k K_k [\mathbf{y}_k - H_k \hat{\mathbf{x}}_k]; \\ K_k = P_k H_k^T [H_k P_k H_k^T + R_k]^{-1}; \\ P_{k+1} = \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T, \end{cases}$$

$$b) \begin{cases} \theta_k = \theta_{k-1} \\ y_k = \psi_k^T \theta_{k-1} + e_k \end{cases}$$

Parâmetros

$$\begin{cases} K_k = P_{k-1} \psi_k [\psi_k^T P_{k-1} \psi_k + R_k]^{-1} \\ \hat{\theta}_k = \hat{\theta}_{k-1} + K_k [y_k - \psi_k^T \hat{\theta}_{k-1}] \\ P_k = P_{k-1} - K_k \psi_k^T P_{k-1} \end{cases}$$

$$c) \begin{aligned} \mathbf{x}(k) &= [\Phi \ \Gamma] \begin{bmatrix} \mathbf{x}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix} \\ &= \Delta^T \mathbf{m}(k-1)^T. \end{aligned}$$

$$\mathbf{x}(k)^T = \mathbf{m}(k-1) \Delta$$

$$= [\mathbf{x}(k-1)^T \ \mathbf{u}(k-1)^T] \begin{bmatrix} \Phi^T \\ \Gamma^T \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}(1)^T \\ \mathbf{x}(2)^T \\ \vdots \\ \mathbf{x}(N)^T \end{bmatrix} = \begin{bmatrix} \mathbf{m}(0) \\ \mathbf{m}(1) \\ \vdots \\ \mathbf{m}(N-1) \end{bmatrix} \Delta$$

$$X_N = M \Delta.$$

Matrizes Est.

$$\begin{cases} K_k = \frac{P_{k-1} \mathbf{m}(k-1)^T}{1 + \mathbf{m}(k-1)^T P_{k-1} \mathbf{m}(k-1)}; \\ \hat{\Delta}_k = \hat{\Delta}_{k-1} + K_k [\mathbf{x}^T(k) - \mathbf{m}(k-1) \hat{\Delta}_{k-1}] \\ P_k = P_{k-1} - \frac{P_{k-1} \mathbf{m}(k-1)^T \mathbf{m}(k-1) P_{k-1}^T}{1 + \mathbf{m}(k-1)^T P_{k-1} \mathbf{m}(k-1)}. \end{cases}$$

Estimação das Matrizes de Estado

Estimação conjunta de estados e parâmetros

Matriz aumentada:

$$\begin{bmatrix} \hat{\mathbf{x}}_{k+1}^+ \\ \hat{\boldsymbol{\theta}}_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi_k & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_k^+ \\ \hat{\boldsymbol{\theta}}_k \end{bmatrix} + \begin{bmatrix} \Gamma_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_k \\ \mathbf{0} \end{bmatrix}$$
$$\hat{\mathbf{x}}_{k+1}^{*-} = \Phi_k^* \hat{\mathbf{x}}_k^{*-} + \Gamma_k^* \hat{\mathbf{u}}_k^*,$$

Dificuldade:

- Dimensão da Matriz de Covariância P – Estabilidade!!

O Filtro de Kalman Estendido

Modelo não-linear

$$\begin{cases} \mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k), \\ \mathbf{y}_{k+1} = h(\mathbf{x}_{k+1}) + \mathbf{v}_{k+1}, \end{cases}$$

Jacobianas:

$$Df(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$Df(\mathbf{w}) = \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \frac{\partial f_1}{\partial w_2} & \cdots & \frac{\partial f_1}{\partial w_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial w_1} & \frac{\partial f_n}{\partial w_2} & \cdots & \frac{\partial f_n}{\partial w_n} \end{bmatrix}$$

$$\left\{ \begin{aligned} \hat{\mathbf{x}}_{k+1}^- &= f(\hat{\mathbf{x}}_k^+, \mathbf{u}_k); \\ P_{k+1}^- &= Df(\mathbf{x}_k)P_k^+ Df(\hat{\mathbf{x}}_k^+)^T + Df(\mathbf{w}_k)Q_k Df(\mathbf{w}_k)^T; \\ K_{k+1} &= P_{k+1}^- Dh(\mathbf{x}_{k+1})^T [Dh(\mathbf{x}_{k+1}) Dh(\mathbf{x}_{k+1})^T + \\ &\quad + Dh(\mathbf{x}_{k+1}) R_{k+1} Dh(\mathbf{x}_{k+1})^T]^{-1}; \\ \hat{\mathbf{x}}_{k+1}^+ &= \hat{\mathbf{x}}_{k+1}^- + K_{k+1} [\mathbf{y}_{k+1} - h(\hat{\mathbf{x}}_{k+1}^-)]; \\ P_{k+1}^+ &= P_{k+1}^- - K_{k+1} Dh(\mathbf{x}_{k+1}) P_{k+1}^- \end{aligned} \right.$$

$$Df(\mathbf{x}_k) = Df(\mathbf{x}_k)|_{\mathbf{x}_k = (\hat{\mathbf{x}}_k^+, \mathbf{u}_k, 0)}$$

$$Df(\mathbf{w}_k) = Df(\mathbf{w}_k)|_{\mathbf{x}_k = (\hat{\mathbf{x}}_k^+, \mathbf{u}_k, 0)}$$

$$Dh(\mathbf{x}_{k+1}) = Dh(\mathbf{x}_{k+1})|_{\mathbf{x}_{k+1} = \hat{\mathbf{x}}_{k+1}^-}$$