Robust Stability of Networked Control Systems

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***Abstract*— In this paper, a robust stability criterion is proposed for NCSs liable to model uncertainties and time-varying delays. The analysis concerns the establishment of a maximum allowable delay bound for continous time NCSs under parameter uncertainties. The new proposed criterion is based on the solution of a set of Linear Matrix Inequalities (LMIs). Numerical examples are given to validate the theoretical result.**

***Index Terms*— Networked control systems, stability criterion, robust stability, LMI, Lyapunov function**

I. INTRODUCTION

NETWORKED control systems (NCSs) are closed-loop control systems that operate over a data communication network [1]–[8]. The main feature of this class of systems is that its elements (plant, controller, actuators and sensors) are linked together throughout a network and the information is exchanged in the form of data packets. Several aplications can be connect with networked control systems. For instance, the work in [9] is concerned in distributed irrigation systems to achieve efficient water management in semiarid and arid areas. In [10], NCS-controllers are used for ambient intelligence networks, i.e., systems that are based on low-energy and low-performance nodes connected by a wireless network. For other applications, see, for instance, the references of [6].

Networked control systems raise new challenges when compared with traditional control systems. Due to the communication network, the presence of transmission delays and packet dropout are unavoidable features that can degrade the system's performance and even destabilize the system [11], [12]. On the other hand, the presence of a communication network brings greater flexibility and higher reliability. Furthermore, the introduction of serial communication networks raises high system testability and resource utilization, as well as low cost, space, power and wiring requirements [13], [14].

One of the most basic issues in the field of NCS is the analysis of stability. The pioneer contribution is by Halevi *et al.* [15], where a discrete-time model is presented and the stability is analyzed for systems with constant and periodic delays. In [16] and [17], a continuous-time representation with a zero-order-hold controller is proposed. Kim *et al.* [18] propose a method to obtain a time-delay upper bound for a given network schedule method. The paper [19] studies the problem of stability analysis for continuous-time networked control systems, concerning the investigation of the delay-dependent stability problem by choosing an appropriate type of Lyapunov function candidate and solving a set of LMIs.

Several works have been devoted to the development of methods for delay-dependent robust stability. In practice, it is very difficult to obtain an exact mathematical model due to environmental noise or slowly varying parameters. Therefore, the NCSs almost inevitably present some uncertainties [20]. In [21], a discrete-time state-space model is considered and the condition for robust asymptotic stability is presented in terms of LMIs. The work in [22] takes into account robust stability and stabilization problems that are solved guarantying generalized quadratic stability and generalized quadratic stabilization. In [23], a robust observer-based controller is designed for the problem of congestion control and the feedback control law is obtained using a linear matrix inequality (LMI).

The criterion presented by Zhu *et al.* [19] distinguishes itself from the others in the sense that its network-induced delay maximum upper bound is less conservative. Nevertheless, the work presented in [19] does not handle system's parameters uncertainties.

In this paper, we present a robust delay-dependent stability criterion for uncertain time-varying networked control systems. Following [19], the derivative character of the delay is considered and the results obtained here are shown to be less conservative than the ones in [20].

This paper is organized as follows. Section II presents the system description and preliminaries, taking into account network-induced delay and packet loss features. In Section III, a new robust criterion for stability analysis is proposed, which is obtained by solving a set of LMI's. Numerical examples are given in Section IV, followed by the conclusions, which are presented in Section V.

II. SYSTEM DESCRIPTION

A closed-loop NCS with the possibility of dropping data packets and disordering can be described as shown in Fig. 1. Such NCS is composed of a plant G_p, a controller module G_c and a common network. The plant G_p includes one sensor module and one actuator module. All modules (sensor, actuator and controller) have a network element (sender and/or receiver). The sender element transmits data packets through the network and the receiver element acquires them. Single packet transmission is considered, i.e., all data sent or received are assembled together into one network packet and transmitted at the same time.

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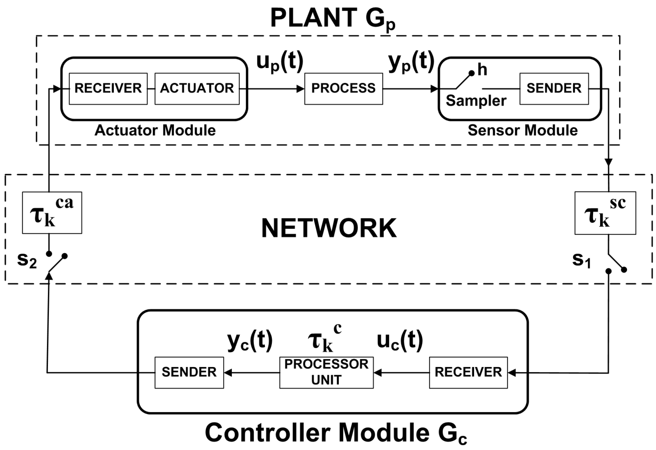

Fig. 1. A NCS with data packet dropout and transmission delays.

Throughout this paper, we assume that the sensor module is *clock-driven* with sampling period h. The controller and actuator modules are *event-driven*. In the event of packet disordering, the actuator module uses the latest available control input.

In the system's modelling, the following delays are considered:

- τ_k^{sc}: delay from sensor to controller module for the kth network packet;
- τ_k^c: computation delay for the kth network packet;
- τ_k^{ca}: delay from controller to actuator module for the kth network packet;
- τ_k: total delay from sensor to actuator module for the kth network packet.

The switches S_1 and S_2 in Fig. 1 model the possibility of packet loss. In the closed position, packets are able to reach their destinies. Otherwise, they are lost.

A. Plant's model

The plant's model presented here is

$$\dot{x}_p(t) = A_p x_p(t) + B_p u_p(t), \quad (1)$$
$$y_p(t) = C_p x_p(t), \quad (2)$$

where $x_p(t) \in \mathbb{R}^{n_p}$ is the plant's state vector, $u_p(t) \in \mathbb{R}^m$ and $y_p(t) \in \mathbb{R}^r$ are the plant's input and output vectors, respectively. The matrices A_p, B_p and C_p are considered not exactly known, but belonging to bounded sets: $A_p \in \mathscr{A}_p \subset \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathscr{B}_p \subset \mathbb{R}^{n_p \times m}$ and $C_p \in \mathscr{C}_p \subset \mathbb{R}^{r \times n_p}$.

B. System's model

Following Fig. 2, the sensor module samples data from the plant at instants nh, where h is the sampling period and $n \in \mathbb{N}^*$. The integers i_k, $k \in \mathbb{N}^*$, denote the nth sample number which is carried by the kth received network packet at the actuator's input.

Remark 1 If $\{i_1, i_2, \ldots, i_n, \ldots\} = \{1, 2, \ldots, n, \ldots\}$, then no packet dropout or disordering occured in the transmission. However, if $i_{k+1} \neq i_k + 1$, then a transmission failure occured.

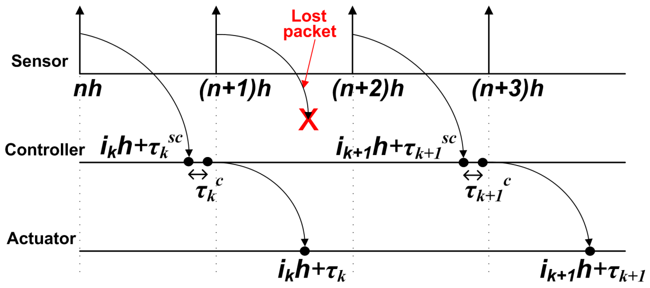

Fig. 2. Time diagram for network-induced delays.

Considering the communication delay from sensor to controller, the controller's input $u_c(t)$ can be described as

$$u_c(t) = y_p(i_k h) = C_p x_p(i_k h), \forall\ k \in \mathbb{N}^*, \quad (3)$$

where y_p is the plant's output, $t \in [i_k h + \tau_k^{sc}, i_{k+1}h + \tau_{k+1}^{sc})$, and $0 \leq \tau_k^{sc} \leq \tau_k^{sc,max}$.

Here we use a proportional control law with constant feedback gain matrix. So, The plant's input $u_p(t)$ from (1) can be described as

$$\begin{aligned} u_p(t) &= y_c(i_k h + \tau_k^{sc} + \tau_k^c + \tau_k^{ca}) \\ &= K C_p x_p(i_k h), \end{aligned} \quad (4)$$

where y_c is the controller's output, $t \in [i_k h + \tau_k, i_{k+1}h + \tau_{k+1})$, and $0 \leq \tau_k \leq \tau_k^{max}$, $\forall k \in \mathbb{N}^*$. Hence, $\tau_k = \tau_k^{sc} + \tau_k^c + \tau_k^{ca}$ and $\tau_k^{max} = \tau_k^{sc,max} + \tau_k^{c,max} + \tau_k^{ca,max}$. Similarly to [24], we assume the existence of constants η and τ, $0 \leq \tau \leq \eta$, such that

$$\begin{aligned} &(i_{k+1} - i_k)h + \tau_{k+1} \leq \eta, \\ &\tau \leq \tau_k, \qquad \forall k \in \mathbb{N}^*. \end{aligned}$$

The element η denote upper bound of the total network-induced delay, involving both transmission delays and packet dropouts. Actually, η limits the total network-induced delay. The term τ denotes lower bounds and has a analogous definition.

Using (1)-(4) the system's model can be described as:

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)), \quad (5)$$
$$x(t) = \phi(t), t \in [t_1 - \eta,\ t_1], \quad (6)$$
$$\tau \leq d(t) \leq \eta, \quad (7)$$
$$d(t) = t - i_k h, \quad t \in [i_k h + \tau_k, i_{k+1}h + \tau_{k+1}), \quad (8)$$

where t_1 denotes the instant that the actuator receives the first control signal and $A_d = B_p K C_p$.

The function $d(t)$ is the time-varying delay from sensor to actuator module. Similarly to [19], we make use of $\dot{d}(t) = 1$ and consider this property for the stability analysis of NCSs.

The equation (5) in closed-loop NCS represented by equations (5)-(8) with uncertainties and time-varying can be rewritten like:

$$\dot{x}(t) = (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t - d(t)) \quad (9)$$

The uncertainties ΔA e ΔA_d are time-varying matrices with appropriate dimensions, which are defined as follows:

$$\Delta A = M_A F_A N_A, \quad (10)$$
$$\Delta A_d = M_{Ad} F_{Ad} N_{Ad} \quad (11)$$

where M_A, N_A, M_{Ad} and N_{Ad} are known real constant matrices with appropriate dimensions and F_A, F_{Ad} represent unknown time-varying matrices that are bounded by $F_A^T F_A \leq I$ and $F_{Ad}^T F_{Ad} \leq I$

III. STABILITY ANALYSIS

This subsection presents a new robust stability criterion for NCSs with model uncertainties. The resulting theorem, written in the form of a set of LMIs, is based on the Lyapunov function candidate

$$V(t) = \sum_{i=1}^{3} V_i(t), \tag{12}$$

where

$$\begin{aligned} V_1(t) &= x^T(t)Px(t), \\ V_2(t) &= \int_{t-\tau}^{t} \left[x(s)^T Q_1 x(s)\right] ds + \int_{t-\eta}^{t} \left[x(s)^T Q_2 x(s)\right] ds \\ &\quad + \int_{t-\alpha d(t)}^{t} \left[x(s)^T Q_3 x(s)\right] ds, \qquad (13) \\ V_3(t) &= \int_{-\eta}^{0} \int_{t+\beta}^{t} \left[\dot{x}(s)^T Z_1 \dot{x}(s)\right] ds d\beta \\ &\quad + \int_{-\eta}^{-\tau} \int_{t+\beta}^{t} \left[\dot{x}(s)^T Z_2 \dot{x}(s)\right] ds d\beta, \end{aligned}$$

and matrices $P = P^T > 0,\ Q_i = Q_i^T \geq 0,\ Z_j = Z_j^T > 0,\ i \in \{1,2,3\},\ j \in \{1,2\}$.

Remark 2 Similarly to [19], the derivative character of the time-varying delay function $d(t)$ can be taken into account through the element α considered in (13).

Throughout this subsection, the following results will be useful to derive sufficient conditions for the NCS's stability.

Lemma 1 *For any constants τ and η and matrix M of appropriate dimensions, the following equality holds:*

$$\frac{d}{dt}\left[\int_{-\eta}^{-\tau}\int_{t+\beta}^{t} \left[\dot{x}^T(s) M \dot{x}(s)\right] ds d\beta\right] = (\eta - \tau)\dot{x}^T(t) M \dot{x}(t) - \int_{t-\eta}^{t-\tau} \dot{x}^T(s) M \dot{x}(s) ds$$

Remark 3 Lemma 1 is a simple extension of Liebniz integral rule.

Lemma 2 ([19], [25], [26]) *For given scalars r_1, r_2 and matrix $M \in \mathbb{R}^{m\times m}$ such that $(r_2 - r_1) > 0$ and $M = M^T > 0$, if choosing a vectorial function $x : [r_1, r_2] \longrightarrow \mathbb{R}^m$ yields:*

$$\int_{r_1}^{r_2} x^T(\beta) M x(\beta) d\beta \geq \frac{1}{(r_2 - r_1)} \left(\int_{r_1}^{r_2} x(\beta) d\beta\right)^T M \left(\int_{r_1}^{r_2} x(\beta) d\beta\right)$$

Lemma 3 ([20], [27]) *For any vectors $x, y \in \mathbb{R}^n$ and appropriate dimensions real matrices A, D, F, E, $P > 0$ and any scalar $\varepsilon > 0$, if $F^T F \leq I$, then*

(i) $DFE + E^T F^T D^T \leq \varepsilon^{-1} DD^T + \varepsilon EE^T$,

(ii) *If $P - \varepsilon DD^T > 0$, then* $(A + DFE)^T P^{-1}(A + DFE) \leq A^T \left(P - \varepsilon DD^T\right)^{-1} A + \varepsilon^{-1} E^T E$.

Lemma 4 ([28]) *For matrices A, B, Q and $X = X^T$ of appropriate dimensions, the Riccati inequality*

$$A^T X + XA + XBB^T X + Q < 0$$

can be rewritten as the LMI

$$\begin{bmatrix} A^T X + XA + Q & XB \\ B^T X & -I \end{bmatrix} < 0.$$

Lemma 5 *For matrices of appropriate dimensions, the following statements are equivalent:*

$$(i) \begin{bmatrix} M & \alpha & R \\ \alpha^T & Z & \beta \\ R^T & \beta^T & Q \end{bmatrix} > 0,$$

$$(ii) \begin{bmatrix} M - RQ^{-1}R^T & \alpha - RQ^{-1}\beta^T \\ \alpha^T - \beta Q^{-1} R^T & Z - \beta Q^{-1}\beta^T \end{bmatrix} > 0, Q > 0$$

Proof:

Taking the Schur's complement of Q in (i) yields

$$\begin{aligned} &\begin{bmatrix} M & \alpha \\ \alpha^T & Z \end{bmatrix} - \begin{bmatrix} R \\ \beta \end{bmatrix} Q^{-1} \begin{bmatrix} R^T & \beta^T \end{bmatrix} > 0 \\ \Leftrightarrow &\begin{bmatrix} M - RQ^{-1}R^T & \alpha - RQ^{-1}\beta^T \\ \alpha^T - \beta Q^{-1}R^T & Z - \beta Q^{-1}\beta^T \end{bmatrix} > 0. \end{aligned}$$

■

Using (12) as the Lyapunov function candidate for the stability analysis of the NCS described by (9)-(11), a delay-dependent asymptotical stability criterion for NCSs with model uncertainties is derived as follows.

Theorem 1 *For given scalars $0 \leq \tau < \eta$, $0 < \alpha < 1$, $\beta_A > 0$, $\beta_{Ad} > 0$ and $\varepsilon > 0$ such that*

$$\begin{bmatrix} M_A^T U M_A & M_A^T U M_{Ad} \\ M_{Ad}^T U M_A & M_{Ad}^T U M_{Ad} \end{bmatrix} - \varepsilon^{-1} I < 0,$$

the NCS described by (9)-(11) is asymptotically stable if there exist matrices $P = P^T > 0$, $Q_i = Q_i^T \geq 0$, $Z_j = Z_j^T > 0$, $i \in \{1,2,3\},\ j \in \{1,2\}$ such that

$$\begin{bmatrix} G^T X + XG + K & XB \\ B^T X & -I \end{bmatrix} < 0. \tag{14}$$

holds, where

$$
\begin{aligned}
U &= \eta Z_1 + (\eta-\tau) Z_2, \\
R_q &= \begin{bmatrix} A^T U M_A & A^T U M_{Ad} \\ A_d^T U M_A & A_d^T U M_{Ad} \end{bmatrix}, \\
Q_q &= \left[-\varepsilon^{-1} I + \begin{bmatrix} M_A^T \\ M_{Ad}^T \end{bmatrix} U \begin{bmatrix} M_A & M_{Ad} \end{bmatrix} \right], \\
A_q &= \begin{bmatrix} 0 & 0 & \frac{1}{\alpha\eta} Z_1 \\ \frac{1}{\eta-\tau} Z_2 & \frac{1}{\eta-\tau}(Z_1+Z_2) & \frac{1}{(1-\alpha)\eta} Z_1 \end{bmatrix}, \\
Z_q &= \begin{bmatrix} Z_{q11} & 0 & 0 \\ 0 & Z_{q22} & 0 \\ 0 & 0 & Z_{q33} \end{bmatrix}, \\
Z_{q11} &= -Q_1 - \frac{1}{\eta-\tau} Z_2 \\
Z_{q22} &= -Q_2 - \frac{1}{\eta-\tau} (Z_1+Z_2) \\
Z_{q33} &= -(1-\alpha) Q_3 - \frac{1}{\alpha\eta} Z_1 - \frac{1}{(1-\alpha)\eta} Z_1
\end{aligned}
$$

$$
\begin{aligned}
X &= \begin{bmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
G &= \begin{bmatrix} A & A_d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B &= \begin{bmatrix} M_A\sqrt{\beta_A} & M_{Ad}\sqrt{\beta_{Ad}^{-1}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
K &= \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{12}^T & K_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_q & R_q \\ A_q^T & Z_q & 0 \\ R_q^T & 0 & Q_q \end{bmatrix} \\
K_{11} &= A^T U A + \varepsilon^{-1} N_A^T N_A - \frac{1}{\alpha\eta} Z_1 \\
&\quad + Q_1 + Q_2 + Q_3 + \beta_A^{-1} N_A^T N_A \\
K_{12} &= A^T U A_d \\
K_{22} &= A_d^T U A_d + \varepsilon^{-1} N_{Ad}^T N_{Ad} - \frac{1}{(1-\alpha)\eta} Z_1 \\
&\quad - \frac{1}{\eta-\tau} Z_2 - \frac{1}{\eta-\tau}(Z_1+Z_2) + \beta_{Ad} N_{Ad}^T N_{Ad}
\end{aligned}
$$

Proof: Taking the time derivative of the Lyapunov function candidate (12) yields

$$
\begin{aligned}
\dot{V}_1(t) &= \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t), \qquad (15) \\
\dot{V}_2(t) &= x^T(t) Q_1 x(t) - x^T(t-\tau) Q_1 x(t-\tau) + x^T(t) Q_2 x(t) \\
&\quad - x^T(t-\eta) Q_2 x(t-\eta) + x^T(t) Q_3 x(t) \\
&\quad - (1-\alpha) x^T(t-\alpha d(t)) Q_3 x(t-\alpha d(t)) \qquad (16) \\
\dot{V}_3(t) &= \frac{d}{dt}\left[\int_{-\eta}^{0} \int_{t+\beta}^{t} \left[\dot{x}(s)^T Z_1 \dot{x}(s) \right] ds d\beta \right] \\
&\quad + \frac{d}{dt}\left[\int_{-\eta}^{-\tau} \int_{t+\beta}^{t} \left[\dot{x}(s)^T Z_2 \dot{x}(s) \right] ds d\beta \right]. \qquad (17)
\end{aligned}
$$

From Lemma 1, (17) can be written as

$$
\begin{aligned}
\dot{V}_3(t) &= \dot{x}(t)^T (\eta Z_1 + (\eta-\tau) Z_2) \dot{x}(t) \\
&- \int_{t-\alpha d(t)}^{t} \left[\dot{x}(s)^T Z_1 \dot{x}(s) \right] ds - \int_{t-d(t)}^{t-\alpha d(t)} \left[\dot{x}(s)^T Z_1 \dot{x}(s) \right] ds \\
&- \int_{t-\eta}^{t-d(t)} \left[\dot{x}(s)^T (Z_1+Z_2) \dot{x}(s) \right] ds - \int_{t-d(t)}^{t-\tau} \left[\dot{x}(s)^T Z_2 \dot{x}(s) \right] ds.
\end{aligned} \qquad (18)
$$

Applying Lemma 2 to (18) yields

$$
\begin{aligned}
&\dot{V}_3(t) \leq \\
&\dot{x}(t)^T U \dot{x}(t) - \frac{1}{\alpha\eta} [x(t) - x(t-\alpha d(t))]^T Z_1 [x(t) - x(t-\alpha d(t))] \\
&- \frac{1}{(1-\alpha)\eta} [x(t-\alpha d(t)) - x(t-d(t))]^T Z_1 [x(t-\alpha d(t)) - x(t-d(t))] \\
&- \frac{1}{\eta-\tau} [x(t-\tau) - x(t-d(t))]^T Z_2 [x(t-\tau) - x(t-d(t))] \\
&- \frac{1}{\eta-\tau} [x(t-d(t)) - x(t-\eta)]^T (Z_1+Z_2) [x(t-d(t)) - x(t-\eta)]. \qquad (19)
\end{aligned}
$$

By (9)-(11), (15) can be written as

$$
\dot{V}_1(t) = \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}^T H \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}, \qquad (20)
$$

where

$$
H = \begin{bmatrix} (A+\Delta A)^T P + P(A+\Delta A) & P(A_d+\Delta A_d) \\ (A_d+\Delta A_d)^T P & 0 \end{bmatrix}.
$$

Applying Lemma 3 (i) to (20), one can obtain

$$
\dot{V}_1(t) \leq \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}^T L \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}, \qquad (21)
$$

where

$$
\begin{aligned}
L &= \begin{bmatrix} L_{11} & PA_d \\ A_d^T P & \beta_{Ad} N_{Ad}^T N_{Ad} \end{bmatrix}, \\
L_{11} &= A^T P + PA + \beta_A^{-1} N_A^T N_A \\
&\quad + P\left(\beta_A M_A M_A^T + \beta_{Ad}^{-1} M_{Ad} M_{Ad}^T \right) P.
\end{aligned}
$$

By (9), the term $\dot{x}^T(t) U \dot{x}(t)$ in (19) may be written as:

$$
\dot{x}^T(t) U \dot{x}(t) = \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}^T T \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix},
$$

where,

$$
\begin{aligned}
T &= \begin{bmatrix} (A+M_A F_A(t) N_A)^T \\ (A_d+M_{Ad} F_{Ad}(t) N_{Ad})^T \end{bmatrix} U \begin{bmatrix} (A+M_A F_A(t) N_A)^T \\ (A_d+M_{Ad} F_{Ad}(t) N_{Ad})^T \end{bmatrix}^T \\
&= \left(\begin{bmatrix} A^T \\ A_d^T \end{bmatrix} + \begin{bmatrix} N_A^T & 0 \\ 0 & N_{Ad}^T \end{bmatrix} \begin{bmatrix} F_A^T & 0 \\ 0 & F_{Ad}^T \end{bmatrix} \begin{bmatrix} M_A^T \\ M_{Ad}^T \end{bmatrix} \right) U \\
&\quad \left(\begin{bmatrix} A & A_d \end{bmatrix} + \begin{bmatrix} M_A & M_{Ad} \end{bmatrix} \begin{bmatrix} F_A & 0 \\ 0 & F_{Ad} \end{bmatrix} \begin{bmatrix} N_A & 0 \\ 0 & N_{Ad} \end{bmatrix} \right)
\end{aligned} \qquad (22)
$$

Applying Lemma 3 (ii) to (22) yields

$$
\begin{aligned}
T \leq & \begin{bmatrix} A^T \\ A_d^T \end{bmatrix} \left(U^{-1} - \varepsilon \begin{bmatrix} M_A & M_{Ad} \end{bmatrix} \begin{bmatrix} M_A^T \\ M_{Ad}^T \end{bmatrix} \right)^{-1} \begin{bmatrix} A & A_d \end{bmatrix} + \\
& \varepsilon^{-1} \begin{bmatrix} N_A^T & 0 \\ 0 & N_{Ad}^T \end{bmatrix} \begin{bmatrix} N_A & 0 \\ 0 & N_{Ad} \end{bmatrix}.
\end{aligned}
$$

Then, using the well known Woodbury matrix identity yields

$$T \leq T_f = \begin{bmatrix} A^T UA + \varepsilon^{-1} N_A^T N_A & A^T UA_d \\ A_d^T UA & A_d^T UA_d + \varepsilon^{-1} N_{Ad}^T N_{Ad} \end{bmatrix} - \begin{bmatrix} A^T \\ A_d^T \end{bmatrix} U \begin{bmatrix} M_A^T \\ M_{Ad}^T \end{bmatrix}^T \left(\varepsilon^{-1} I - \begin{bmatrix} M_A^T \\ M_{Ad}^T \end{bmatrix} U \begin{bmatrix} M_A & M_{Ad} \end{bmatrix} \right)^{-1} \begin{bmatrix} M_A^T \\ M_{Ad}^T \end{bmatrix} U \begin{bmatrix} A^T \\ A_d^T \end{bmatrix}^T .$$

Therefore,

$$\dot{V}_3 \leq \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}^T T_f \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix} . \tag{23}$$

Denoting

$$\delta^T = \left[x^T(t) x^T(t-d(t)) x^T(t-\tau) x^T(t-\eta) x^T(t-\alpha d(t)) \right],$$

the combination of (16), (21) and (23) yields the stability condition

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \leq \begin{bmatrix} M_q - R_q Q_q^{-1} R_q^T & A_q \\ A_q^T & Z_q \end{bmatrix} < 0, \tag{24}$$

$$\begin{aligned} M_q &= \begin{bmatrix} M_{11} & A^T UA_d + PA_d \\ A_d^T UA + A_d^T P & M_{22} \end{bmatrix}, \\ M_{11} &= A^T UA + \varepsilon^{-1} N_A^T N_A - \frac{1}{\alpha\eta} Z_1 + Q_1 + Q_2 + Q_3 \\ &\quad + A^T P + PA + \beta_A^{-1} N_A^T N_A \\ &\quad + P \left[\beta_A M_A M_A^T + \beta_{Ad}^{-1} M_{Ad} M_{Ad}^T \right] P, \\ M_{22} &= A_d^T UA_d + \varepsilon^{-1} N_{Ad}^T N_{Ad} - \frac{1}{(1-\alpha)\eta} Z_1 - \frac{1}{\eta - \tau} Z_2 \\ &\quad - \frac{1}{\eta - \tau}(Z_1 + Z_2) + \beta_{Ad} N_{Ad}^T N_{Ad} . \end{aligned}$$

From Lemma 5, with identities $\alpha \leftarrow A_q$ and $\beta \leftarrow 0$, (24) can be written as

$$\begin{bmatrix} M_q & A_q & R_q \\ A_q^T & Z_q & 0 \\ R_q^T & 0 & Q \end{bmatrix} < 0. \tag{25}$$

Finally, applying Lemma 4 to (25) to eliminate quadratic terms yields (14), completing the proof.

■

Remark 4 Many delay-dependent stability criteria for NCSs, like the ones presented in [3], [18], [19], [24], [26], consider only systems without model uncertainties. This assumption, although satisfactory for some cases, restricts the use of robust theory and excludes from analysis possible uncertainties that may arise. Therefore, Theorem 1 is more general, since it deals with models with or without uncertainties. Indeed, if one uses Theorem 1 with a NCS model that has no uncertainties, it yields the same stability criterion presented in [19]. This can be seen, after some algebra, considering $\Delta A = 0$ and $\Delta A_d = 0$ in (14).

TABLE I

MAXIMUM UPPER BOUND DELAY FOR DIFFERENTS VALUES OF τ

	$\tau = 0.1$s	$\tau = 0.2$s	$\tau = 0.3$s	$\tau = 0.4$s
η	0.6977s	0.7133s	0.7336s	0.7590

IV. NUMERICAL EXAMPLES

In this section, two examples are presented in order to confirm the validity of the proposed criterion. The first one demonstrates the possibility of applying Theorem 1 to a NCS with abscence of uncertainties. The second illustrates the effectiveness of our criterion by establishing a less conservative upper bound delay to the NCS example proposed in [20].

Example 1 Consider the NCS (9)-(11) with no uncertainties [19]:

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \ A_d = \begin{bmatrix} -1.4 & 0 \\ -0.8 & -1.5 \end{bmatrix}, \ \Delta A = 0, \ \Delta A_d = 0,$$

From Theorem 1, with $\alpha = 0.75$ and $\tau = 0.4$ s, the maximum upper bound value for the total delay, η, which mantains the system stability is 1.17 s. The result is the same as the obtained in [19]. Furthermore, this result is less conservative than the values obtained in [24] (1.13 s) and [26] (1.16 s), which endorses the effectiveness of the proposed criterion.

Example 2 For this example, we consider the NCS with uncertainties and time-varying delays presented in [20], described by:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \ A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

with the following uncertainties:

$$M_A = N_A = \begin{bmatrix} \sqrt{0.3} & 0 \\ 0 & \sqrt{0.2} \end{bmatrix}, \ M_{A_d} = N_{A_d} = \begin{bmatrix} \sqrt{0.2} & 0 \\ 0 & \sqrt{0.3} \end{bmatrix},$$

$$F_A(t) = F_{A_d}(t) = \begin{bmatrix} \cos(t) & 0 \\ 0 & \sin(t) \end{bmatrix}.$$

The maximum upper bound values for the total delay presented by previous authors were: 0.1575 s [29], 0.1575 s [30], 0.2558 s [31], 0.3916 s [32] and 0.6090 s [20]. According to Theorem 1, choosing $\alpha = 0.60$, $\varepsilon = 0.9$, $\beta_A = \beta_{Ad} = 0.8$ and $\tau = 0.0$, we found the NCS system is stable for a delay within the interval of 0 to 0.6847 s. This example illustrates that our stability criterion is less conservative than the other works' criteria. Furthermore, choosing $\tau > 0$, we obtain even better results, as shown on Table I.

V. CONCLUSIONS

This paper deals with the problem of stability analysis of networked control systems with uncertainties and time-varying delays. Based on the solution of a set of LMIs, we propose a new asymptotical stability criterion. The new stability criterion is less conservative and more general than some of the existing results. Numerical results illustrate the effectiveness of the proposed criterion.

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