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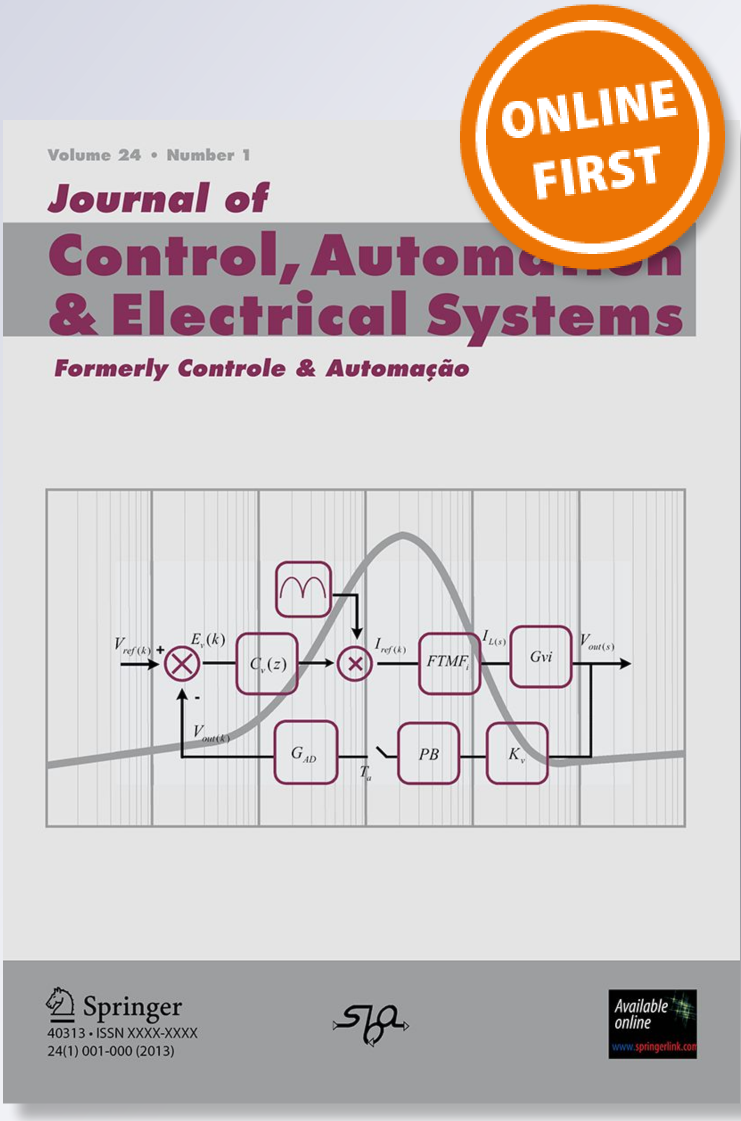

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Abstract This paper addresses the robust stability problem for nonlinear systems subjected to model uncertainties and with time-delay and its derivative varying within intervals. The nonlinear time-delay system is described by a new Takagi-Sugeno fuzzy model consisted of local nonlinear time-delay systems. The new fuzzy model has fewer fuzzy rules than conventional T–S fuzzy models with local linear time-delay systems; therefore, can be more easily derived in practical situations. To reduce conservatism concerning both models, a stability analysis which incorporates state-of-the-art stability techniques with an improved piecewise analysis method, amended with novel delay-interval-dependent terms, is proposed. The proposed analysis, based on a novel fuzzy weighting-dependent Lyapunov-Krasovskii functional, considers that the delay-derivative is either upper and lower bounded, bounded above only, or unbounded, i.e., when no restrictions are cast upon the derivative. Numerical examples are provided to enlighten the importance and the conservatism reduction of the proposed method which outperforms state-of-the-art criteria in time-delay systems literature.

Keywords Time-delay systems · T–S fuzzy models · Delay partitioning · Delay-dependent stability · Fuzzy Lyapunov functional · Nonlinear systems

1 Introduction

A fundamental issue in modern control theory, complex nonlinear systems modeling and stability analysis have been extensively investigated with fuzzy control techniques, particularly within Takagi and Sugeno (1985) modeling framework. In a T–S fuzzy model, local dynamics in different state-space regions are represented by linear models that are smoothly connected by fuzzy membership functions in order to describe the global nonlinear system (Tognetti et al. 2011). It has been proved that the technique can effectively approximate a wide class of nonlinear systems, thus different stability and control methods for nonlinear systems have been developed considering T–S fuzzy models, see (Tanaka and Wang 2001; Teixeira et al. 2000; Arrifano and Oliveira 2004; Tanscheit et al. 2007; Teixeira and Assunção 2007; Andrea et al. 2008, and the references therein). For the stability analysis, the standard quadratic Lyapunov function is the most popular and general approach (Souza et al. 2009). However, to reduce the conservatism, an interesting alternative is to consider fuzzy weighting-dependent strategies that parameterize Lyapunov terms by the same membership functions used to construct the T–S fuzzy model (Mozelli et al. 2008; Tanaka et al. 2007). Recently, T–S fuzzy techniques have been extended to the analysis of nonlinear time-delay systems. Time-delays are often found in practical dynamic systems, and the existence of such phenomena can degrade a system performance and even cause instability. Therefore, the modeling and stability analysis of time-delay systems have emerged as a topic of significant interest in the control

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community, which is highlighted in several surveys, see, e.g., (Dugard and Verriest 1998; Gu et al. 2003).

During the last decade, considerable attention has been devoted to T–S fuzzy time-delay systems stability and control problems, see, e.g., (Chen et al. 2007; Tian and Peng 2006), and the references therein. From existing results in time-delay literature, the works from (Zhao et al. 2011; Yoneyama 2007) must be acknowledged for regarding the membership functions in the stability conditions. Also, (Liu et al. 2010; Peng et al. 2009a; Peng and Han 2011) must be highlighted for their contributions, in the sense of conservatism reduction. Nevertheless, none of the previous results on T–S fuzzy time-delay systems have taken into consideration recent advances from linear delayed systems stability literature. Among recent criteria for linear time-delay systems, two Lyapunov-based strategies must be acknowledged for their significant contributions to delay-dependent stability analysis: the convex analysis technique from (Park and Ko 2007), which amends the widely employed Jensen's inequality; and the piecewise analysis method (PAM), based on concepts similar to the discretized Lyapunov functionals (DLF) technique (Gu et al. 2003), albeit applied to time-varying delays. The PAM relies on the partitioning of the delay range, and has been successfully employed in recent literature. Particularly, (Fridman et al. 2009; Figueredo et al. 2011) also explore the delay derivative lower bound information using specific delay-interval-dependent terms. However, we believe there is still significant room for improvements, for its potential has not yet been fully exploited. Therefore, we propose an improved piecewise analysis method which amends conventional PAM techniques with novel and less-restricted delay-interval-dependent terms, ignored in previous works.

Notwithstanding, in practical situations, it is very hard to obtain treatable T–S fuzzy models with local linear systems that effectively represent complex nonlinear systems. Indeed, conventional T–S fuzzy models for very complex nonlinear systems usually rely on a great set of fuzzy rules, which, apart from being unsuitable for practical implementation, might impose unfeasible conditions to stability and control analyses (Delmotte et al. 2007). A naive solution is to simplify the original nonlinear system to obtain a more convenient T–S fuzzy model, but the obtained conditions might not be valid for the original system (Tanaka and Wang 2001). Recently, Dong et al. (2009) proposed a new class of T–S fuzzy models consisted of local nonlinear systems. The new model is easier to be derived compared to conventional T–S fuzzy model, and has been proved to effectively represent complex nonlinear systems with fewer fuzzy rules (Dong et al. 2009, 2011). According to (Klug and Castelan 2011), the resulting model can also be regarded as a special class of LPV systems with feedback nonlinearities, and whose state-space matrices are assumed to depend on a time-varying parameter (Choi and Park 2003). However, to the best of the authors' knowledge, the advances of considering a more general class of T–S fuzzy local systems have been restricted to delay-free systems, and the analysis for complex nonlinear systems with time-delays still remains challenging. This scenario is the major motivation of the present study.

The present paper brings an important contribution to nonlinear time-delay systems analysis. First, by developing a novel fuzzy weighting-dependent Lyapunov-Krasovskii functional (FWD-LKF) and applying state-of-the-art stability techniques with an improved piecewise analysis method, we are able to derive robust stability conditions for complex T–S fuzzy models with local uncertain nonlinear time-delay systems. Complex time-varying and delay-dependent nonlinearities are assumed to be norm-bounded, satisfying given quadratic constraints. Furthermore, the proposed method if adapted to the stability analysis of conventional T–S fuzzy models with local linear time-delay systems, or even to the analysis of linear time-delay systems, also yields less conservative results than previous criteria in time-delay systems literature. The advantages of the proposed technique is further enlightened with numerical examples that illustrate the conservativeness reduction compared to state-of-the-art criteria, and the effectiveness of the stability analysis for T–S fuzzy models with local nonlinear time-delay systems.

This paper is organized as follows. Section 2 presents the problem formulation, and the new T–S fuzzy model with local nonlinear time-delay systems. In Sect. 3, an LMI-based stability criterion for the novel time-delay T–S fuzzy system is proposed. Numerical examples are given in Sect. 4, followed by the conclusions, which are presented in Sect. 5.

Notations Throughout the paper the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{n\times p}$ defines the set of all $n\times p$ real matrices. The notation $diag\{\cdots\}$ stands for a block-diagonal matrix, $P>0$ means that P is symmetric and positive definite, and the symmetric term in a matrix is denoted by $*$. The notation $A|_{s\to b}$ stands for the limit of a s-dependent matrix A as $s\to b$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2 Problem Formulation and Preliminaries

Consider the following nonlinear time-delay system

$$\dot{x}(t) = f(t, x(t), x(t-d(t))) \tag{1}$$

where $x(t)\in\mathbb{R}^{r_x}$ denote the state vector, and $f:\mathbb{R}_+\times\mathbb{R}^{r_x}\times\mathbb{R}^{r_x}\to\mathbb{R}^{r_x}$ is a sufficiently smooth nonlinear function. In many situations, the nonlinear system (1) may be locally linguistically described by experts using IF–THEN rules. In this context, Takagi and Sugeno (1985) and Sugeno and Kang (1988) proposed an approach to effectively represent the nonlinear process using a class of fuzzy models, referred to as Takagi-Sugeno (T–S) fuzzy models, which are described by a set of IF–THEN rules representing local input-output relations of the nonlinear system. In conventional T–S fuzzy models, the dynamics of each implication are described by a local linear system model. In practice, however, T–S fuzzy models with exclusively local linear systems may require a large set of subsystems and fuzzy rules, which make them hard to analyze and sometimes unsuitable for practical implementation. In this context and in an analogous manner to the work from (Dong et al. 2009) for delay-free systems, we will consider a class of T–S fuzzy models with local nonlinear systems subjected to model uncertainties and with time-varying delay. The idea is to make the description of the nonlinear time-delay system (1) easier, and with fewer fuzzy rules compared to conventional T–S fuzzy model, whereas ensuring conditions for the stability analysis of the nonlinear time-delay system.

From the new T–S fuzzy model, the nonlinear time-delay system (1) can be represented by local uncertain nonlinear time-delay systems with their linguistic descriptions as

Rule i **:** **IF** $\theta_1(t)$ is μ_1^i, $\theta_2(t)$ is μ_2^i, ... and $\theta_p(t)$ is μ_p^i, **THEN**

$$\dot{x}(t) = (A_i + \Delta A_i)x(t) + (A_{di} + \Delta A_{di})x(t-d(t)) + g_i(t, x(t), x(t-d(t))), \quad (2)$$

where $i = 1, 2, \ldots, r$, r is the number of fuzzy IF–THEN rules; μ_j^i $(i = 1, 2, ..., r;\ j = 1, 2, ..., p)$ are the fuzzy sets corresponding to the premise variables $\theta_j(t)$ and the fuzzy rules; A_i and A_{di} are known real constant matrices with appropriate dimension; and $g_i(t, x(t), x(t-d(t)))$ denotes a class of piecewise-continuous nonlinear functions in $t, x(t), x(t-d(t))$,

$$g_i(t, x(t), x(t-d(t))) : \mathbb{R}_+ \times \mathbb{R}^{r_x} \times \mathbb{R}^{r_x} \mapsto \mathbb{R}^{r_x},$$

which are assumed to satisfy the quadratic condition:

$$\|g_i(t, x(t), x(t-d(t)))\|_2^2 \le \alpha_{1i}^2 x^T(t) H_{1i}^T H_{1i} x(t) + \alpha_{2i}^2 x^T(t-d(t)) H_{2i}^T H_{2i} x(t-d(t)), \quad (3)$$

where $i = 1, .., r$, α_{1i}, and α_{2i} are known bounding parameters of $g_i(t, x(t), x(t-d(t)))$, and H_{1i}, H_{2i} are constant matrices with appropriate dimensions.

The systems' uncertainties are assumed time-varying matrices,

$$\left[\,\Delta A_i \ \Delta A_{di}\,\right] = \Xi_{xi}\Delta(t)\left[\,\Xi_{Ai} \ \Xi_{Adi}\,\right], \quad (4)$$

where Ξ_{xi}, Ξ_{Ai}, and Ξ_{Adi} are known constant matrices, and $\Delta(t)$ is an unknown time-varying matrix, which is Lebesgue measurable in t and satisfies $\Delta(t)^T\Delta(t) \le I$.

The continuous function $d(t)$ denotes the time-varying delay which satisfies

$$\tau_{min} \le d(t) \le \tau_{max}, \quad (5a)$$

$$d_{min} \le \dot{d}(t) \le d_{max}, \quad (5b)$$

where the constants $0 \le \tau_{min} \le \tau_{max}$ and $d_{min} \le d_{max}$ denote the bounding parameters of $d(t)$ and $\dot{d}(t)$, respectively. In this paper, we also consider the case when d_{min} is unknown, and when no restrictions are cast upon the delay derivative, i.e., when it is assumed to be fast-varying.

By fuzzy blending, the global dynamics of the T–S fuzzy system (1) can be inferred as follow:

$$\begin{aligned}\dot{x}(t) &= \frac{1}{\sum_{i=1}^r w_i(\theta(t))}\sum_{i=1}^r w_i(\theta(t))[(A_i + \Delta A_i)x(t) \\ &\quad + (A_{di} + \Delta A_{di})x(t-d(t)) + g_i(t, x(t), x(t-d(t)))], \\ &= \left(\overline{A} + \Delta\overline{A}\right)x(t) + \left(\overline{A}_d + \Delta\overline{A}_d\right)x(t-d(t)) \\ &\quad + \overline{g}(t, x(t), x(t-d(t))), \qquad t > 0, \\ x(t) &= \varphi(t), \qquad t \in [-\tau_{max}, 0], \end{aligned} \quad (6)$$

where $\varphi(t)$ describes the state's initial condition,

$$\overline{A} := \sum_{i=1}^r \rho_i(\theta(t))A_i, \qquad \Delta\overline{A} := \sum_{i=1}^r \rho_i(\theta(t))\Delta A_i,$$

$$\overline{A}_d := \sum_{i=1}^r \rho_i(\theta(t))A_{di}, \quad \Delta\overline{A}_d := \sum_{i=1}^r \rho_i(\theta(t))\Delta A_{di},$$

$$\overline{g}(t, x(t), x(t-d(t))) := \sum_{i=1}^r \rho_i(\theta(t)) g_i(t, x(t), x(t-d(t))).$$

and $\theta = \left[\theta_1, \theta_2, \ldots, \theta_p\right]$; $w_i(\theta(t)) \ge 0$ is the membership function with respect to the rule i, $i = 1, 2, \ldots, r$; and $\rho_i(\theta(t)) = \frac{w_i}{\sum_{i=1}^r w_i(\theta(t))}$ is the normalized fuzzy weighting functions satisfying

$$\rho_i(\theta(t)) \ge 0, \qquad \sum_{i=1}^r \rho_i(\theta(t)) = 1, \qquad \sum_{i=1}^r \dot{\rho}_i(\theta(t)) = 0. \quad (7)$$

To make the reading easier, $\rho_i(t)$ denotes $\rho_i(\theta(t))$.

Throughout the next section, the following result will be useful to derive conditions for the establishment of new delay-dependent stability criterion for system (6).

Lemma 1 *(***Jensen's inequality***) For given scalars r_1, r_2 and matrix $M \in \mathbb{R}^{m\times m}$ such that $(r_2 - r_1) \ge 0$ and $M > 0$, and any vectorial function $x : [r_1, r_2] \longrightarrow \mathbb{R}^m$, we have:*

$$(r_2 - r_1)\int_{r_1}^{r_2} x^T(\beta)Mx(\beta)d\beta \ge \left(\int_{r_1}^{r_2} x(\beta)d\beta\right)^T M\left(\int_{r_1}^{r_2} x(\beta)d\beta\right).$$

3 Stability Analysis

This section presents the main result of this paper. We derive conditions under which the T–S fuzzy time-delay system subjected to model uncertainties and with local uncertain nonlinear systems (6) achieves robust asymptotic stability.

First, we shall, similar to (Fridman et al. 2009; Figueredo et al. 2010), divide the delay range $[\tau_{min}, \tau_{max}]$. Here, we will consider two equally spaced subintervals: $[\tau_1, \tau_2]$ and $[\tau_2, \tau_3]$, where $\tau_1 = \tau_{min}$, $\tau_3 = \tau_{max}$, and $\tau_2 = \frac{1}{2}(\tau_{max} - \tau_{min}) + \tau_{min}$. Note that one can consider different partitioning strategies (e.g., Orihuela et al. (2010) allow τ_2 to be anywhere between τ_{min} and τ_{max}). Still, choosing equally subintervals, $\tau_3 - \tau_2 = \tau_2 - \tau_1$, adds more information which is used to obtain less conservative criteria. In this context, we define $\delta_\tau := \tau_2 - \tau_1$, and the delay-interval-dependent indicator function $\chi_{[\tau_1,\tau_2]} : \mathbb{R}_+ \rightarrow \{0, 1\}$, which is assumed to be 1, if $d(t) \in [\tau_1, \tau_2]$, and $\chi_{[\tau_1,\tau_2]} = 0$, otherwise. The indicator function enlightens the piecewise analysis method's main contribution: the establishment of different linear matrix inequalities (LMIs) for each subinterval, reducing the conservatism which arises from the analysis of the delay range $[\tau_{min}, \tau_{max}]$. In this context, the following fuzzy weighting-dependent Lyapunov-Krasovskii candidate is proposed to ensure the robust asymptotic stability of the T–S fuzzy model with local uncertain nonlinear time-delay systems.

$$V(t) = \sum_{j=1}^{5} V_j(t), \tag{8}$$

where

$$\begin{aligned}
V_1(t) &= \chi_{[\tau_1,\tau_2]} x^T(t)\left(\frac{d(t)-\tau_1}{\tau_2-\tau_1}P_m(t) + \frac{\tau_2-d(t)}{\tau_2-\tau_1}P_1(t)\right)x(t) \\
&\quad + (1-\chi_{[\tau_1,\tau_2]})x^T(t)\left(\frac{d(t)-\tau_2}{\tau_3-\tau_2}P_2(t) + \frac{\tau_3-d(t)}{\tau_3-\tau_2}P_m(t)\right)x(t), \\
V_2(t) &= \int_{t-d(t)}^{t-\tau_1} x^T(s)Qx(s)ds, \\
V_3(t) &= \int_{t-\tau_2}^{t-\tau_1} \left[x^T(s)\ x^T(s-\delta_\tau)\right] N \left[x^T(s)\ x^T(s-\delta_\tau)\right]^T ds \\
&\quad + \int_{t-\frac{\tau_1}{2}}^{t} \left[x^T(s)\ x^T(s-\tfrac{\tau_1}{2})\right] M \left[x^T(s)\ x^T(s-\tfrac{\tau_1}{2})\right]^T ds, \\
V_4(t) &= \sum_{k=1}^{2}\left(\frac{\tau_1}{2}\int_{-\frac{k}{2}\tau_1}^{-\frac{k-1}{2}\tau_1}\int_{t+\beta}^{t}\dot{x}^T(s)S_k\dot{x}(s)dsd\beta \right. \\
&\quad \left. + \delta_\tau \int_{-\tau_{k+1}}^{-\tau_k}\int_{t+\beta}^{t}\dot{x}^T(s)Z_k\dot{x}(s)dsd\beta\right) \\
V_5(t) &= \int_{-d(t)}^{0}\int_{t+\beta}^{t}\dot{x}^T(s)(R_1+R_2)\dot{x}(s)dsd\beta + \int_{-\tau_3}^{-d(t)}\int_{t+\beta}^{t}\dot{x}^T(s) \\
&\quad \times (R_3+R_4)\dot{x}(s)dsd\beta + \chi_{[\tau_1,\tau_2]}\int_{-\tau_2}^{-d(t)}\int_{t+\beta}^{t}\dot{x}^T(s)(R_1-R_3)\dot{x}(s) \\
&\quad \times dsd\beta + (1-\chi_{[\tau_1,\tau_2]})\int_{-d(t)}^{-\tau_2}\int_{t+\beta}^{t}\dot{x}^T(s)(R_3-R_1)\dot{x}(s)dsd\beta,
\end{aligned}$$

where $V_1(t)$ introduces the concept of a delay-interval and fuzzy weighting-dependent LKF term, and it is defined with the function matrices

$$P_1(t) = \sum_{j=1}^{r}\rho_j(t)P_{1j}, \quad P_2(t) = \sum_{j=1}^{r}\rho_j(t)P_{2j}, \tag{9}$$

and $P_m(t) = \frac{1}{2}(P_2(t) + P_1(t))$.

Recently, fuzzy weighting-dependent function matrices have been successfully employed for delay-free traditional T–S fuzzy systems in order to reduce the analysis conservatism. Nonetheless, results on time-delay fuzzy systems mostly regard quadratic fuzzy weighting-independent Lyapunov-Krasovskii functionals. Note that it is not trivial to consider FWD-LKF terms, since its derivative relies on the time-derivative information regarding the membership functions, i.e., $\dot{\rho}_i(t)$. In this case, we must consider the following

Assumption 1 The function $\rho_i(\theta(t))$, $i = \{1, \ldots, r\}$, is continuously differentiable in t with $\dot{\rho}_i(\theta(t)) \leq \sigma_i$, where $\sigma_i \geq 0$ are constant known bounding parameters.

Remark that, in practice, the bounds σ_i largely depend on the information of the membership functions, and their estimation may be somewhat difficult, (Souza et al. 2009). In (Tanaka et al. 2007), the authors provide some strategies to regard and estimate these upper bounds. However, in the case of intense variation of the fuzzy weighting functions, where σ_i is either very large or difficult to estimate, $P_k(t)$ may be regarded as parameter-independent constant matrices, i.e., $P_k(t) = P_k$, $k = \{1, 2\}$, such that (8) yields a Lyapunov-Krasovskii functional.

Concerning (8), it is interesting to highlight that the instantaneous energy-like function $x^T(t)Px(t)$ would not be suitable for investigating the nonlinear system (1), since the time-delay system belongs to the infinite-dimensional system class (Gu et al. 2003). In this context, the notion of a delay-dependent Lyapunov functional seems like an interesting choice (Dugard and Verriest 1998). In the absence of the time-delay-interval size, we could only consider terms similar to $V_3(t)$, e.g., $\int_{t-\tau}^{t} x^T(s)Mx(s)ds$, which yields delay-independent stability conditions (Gu et al. 2003). Notwithstanding, taking the time-delay information (5), we are able to consider the LKFs, $V_4(t)$ and $V_2(t)$, which yield delay-dependent and delay-derivative-dependent stability conditions, respectively. A further contribution of the present criterion regards the introduction of the delay-interval-dependent terms $V_1(t)$ and $V_5(t)$ which lead to delay-interval-dependent stability conditions. Note that $V_1(t)$ also regards the fuzzy membership functions, being a delay-interval fuzzy weighting-dependent functional. Therefore, $V(t)$ is also a fuzzy weighting-dependent Lyapunov-Krasovskii functional.

The positiveness of the FWD-LKF (8) is assured for any $x(t)$, if the following conditions hold, $k = \{1, 2\}$,

$$\begin{aligned}
&P_{kj} > 0,\ j = \{1, \ldots, r\},\ S_k > 0,\ Z_k > 0,\ Q \geq 0,\ N \geq 0, \\
&M \geq 0,\ (R_1 + R_2 + S_k) > 0,\ (R_1 + R_2 + Z_k) > 0, \\
&(R_3 + R_4 + Z_k) > 0,\ Z_2 > \frac{1}{\delta_\tau}(R_1 - R_3) > -Z_1.
\end{aligned} \tag{10}$$

Also, the Lyapunov candidate (8) is continuous in t, since

$$\lim_{d(t)\to\tau_2^+} V_1(t) = \lim_{d(t)\to\tau_2^-} V_1(t) = x^T(t)P_m(t)x(t), \quad \text{and},$$

$$\lim_{d(t)\to\tau_2^+} V_5(t) = \lim_{d(t)\to\tau_2^-} V_5(t) = \int_{-\tau_2}^{0}\int_{t+\beta}^{t} \dot{x}^T(s)(R_1+R_2) \times \dot{x}(s)dsd\beta + \int_{-\tau_3}^{-\tau_2}\int_{t+\beta}^{t} \dot{x}^T(s)(R_3+R_4)\dot{x}(s)dsd\beta.$$

The following result based on (8) states a novel robust delay-dependent criterion which, if satisfied, guarantees the asymptotic stability of the T–S fuzzy model described by local uncertain nonlinear time-delay systems, (6).

Theorem 1 *Given scalars τ_{min}, τ_{max}, d_{min}, and d_{max} such that $0 \le \tau_{min} \le \tau_{max}$ and $d_{min} < d_{max}$, and under the Assumption 1, the T–S fuzzy time-delay system (6) described by local systems with time-varying delay satisfying (5), and subjected to model uncertainties and nonlinearities, respectively, satisfying (4) and (3) is robustly asymptotically stable if there exist positive scalars $\epsilon_1, \epsilon_2, \phi_1, \phi_2$, and matrices P_{kj}, S_k, Z_k, $k = \{1, 2\}$, $j = \{1, ..., r\}$, Q, N, M, R_1, R_2, R_3, and R_4 satisfying (10) and free-weighting matrices $\mathcal{F}_{11}, \mathcal{F}_{12} \in \mathbb{R}^{7r_x \times r_x}$, $\mathcal{F}_{21}, \mathcal{F}_{22} \in \mathbb{R}^{7r_x \times 2r_x}$, and $X_\eta \in \mathbb{R}^{r_x \times r_x}$, $\eta = \{1, 2, 3, 4\}$, such that the following LMIs*

$$\begin{matrix} P_{mi} + X_1 \ge 0, & P_{1i} + X_2 \ge 0, \\ P_{2i} + X_3 \ge 0, & P_{mi} + X_4 \ge 0, \end{matrix} \quad i = \{1, \ldots, r\}, \tag{11}$$

$$S_1 + U_1|_{\dot{d}(t)\to d_{min}} > 0, \qquad S_2 + U_1|_{\dot{d}(t)\to d_{min}} > 0, \tag{12a}$$
$$S_1 + U_1|_{\dot{d}(t)\to d_{max}} > 0, \qquad S_2 + U_1|_{\dot{d}(t)\to d_{max}} > 0, \tag{12b}$$

and

$$\Omega_{1k}^{[i]}|_{\dot{d}(t)\to d_{min}} < 0, \qquad \Omega_{2k}^{[i]}|_{\dot{d}(t)\to d_{min}} < 0, \tag{13a}$$
$$\Omega_{1k}^{[i]}|_{\dot{d}(t)\to d_{max}} < 0, \qquad \Omega_{2k}^{[i]}|_{\dot{d}(t)\to d_{max}} < 0, \tag{13b}$$

hold for $P_{mi} = \frac{1}{2}(P_{2i} + P_{1i})$, and $\Omega_{\ell k}^{[i]}$ defined in (14), $\ell, k = \{1, 2\}$, $i = \{1, 2, ..., r\}$.

Moreover, if the above conditions are satisfied with $X_\eta = 0$, $\eta = \{1, 2, 3, 4\}$, and $P_{ki} = P_k > 0$, $k = \{1, 2\}$, $i = \{1, 2, ..., r\}$, then (8) loses the fuzzy weighting-dependence and (6) is robustly asymptotically stable regardless from Assumption 1.

$$\Omega_{\ell k}^{[i]} = \begin{bmatrix} \Pi_\ell^{[i]} + \Psi_\ell^{[i]}|_{d(t)\to\tau_{(\ell+k-1)}} & \begin{bmatrix} \delta_\tau \mathcal{F}_{2\ell} J_k & \mathcal{F}_{1\ell}\Xi_{xi} & \Gamma_\Xi^{[i]} & \mathcal{F}_{1\ell} \end{bmatrix} \\ * & -diag\{(\delta_\tau \Lambda_{\ell k}), \epsilon_\ell I, \epsilon_\ell I, \phi_\ell I\} \end{bmatrix}, \qquad for\ \ell, k = \{1, 2\}, \tag{14}$$

where $J_1 = \begin{bmatrix} 0 & I \end{bmatrix}^T$, $J_2 = \begin{bmatrix} I & 0 \end{bmatrix}^T$, and

$$U_R = R_1 + \dot{d}(t)R_4 + \left(1 - \dot{d}(t)\right)R_2, \qquad U_1 = \frac{2}{\tau_1}U_R,\ \text{if}\ \tau_1 > 0, \text{ or } U_1 = 0 \text{ otherwise;}$$

$$\Gamma_1 = J_2(\mathbb{I}_2 - \mathbb{I}_5) + J_1(\mathbb{I}_6 - \mathbb{I}_2), \qquad \Gamma_2 = J_2(\mathbb{I}_2 - \mathbb{I}_6) + J_1(\mathbb{I}_7 - \mathbb{I}_2),$$

$$\widetilde{P}_1^{[i]} = \frac{d(t) - \tau_1}{\delta_\tau}P_{mi} + \frac{\tau_2 - d(t)}{\delta_\tau}P_{1i}, \qquad \widetilde{P}_2^{[i]} = \frac{d(t) - \tau_2}{\delta_\tau}P_{2i} + \frac{\tau_3 - d(t)}{\delta_\tau}P_{mi},$$

$$\widetilde{R} = (\tau_3 - d(t))R_4 + d(t)R_2, \qquad \Gamma_\Xi^{[i]} = \epsilon_\ell\left(\mathbb{I}_1\Xi_{Ai}^T + \mathbb{I}_2\Xi_{Adi}^T\right),$$

$$\Lambda_{11} = \delta_\tau Z_1 + R_1 + R_4, \qquad \Lambda_{12} = \delta_\tau Z_1 + U_R,$$

$$\Lambda_{21} = \delta_\tau Z_2 + R_3 + R_4, \qquad \Lambda_{22} = \delta_\tau Z_2 + U_R + (R_3 - R1),$$

$$\Psi_1^{[i]} = \widetilde{\Psi}^{[i]} - (\mathbb{I}_6 - \mathbb{I}_7)\frac{1}{\delta_\tau}\Lambda_{21}(\mathbb{I}_6 - \mathbb{I}_7)^T + \mathbb{I}_3\widetilde{R}\mathbb{I}_3^T + \mathbb{I}_1\widetilde{P}_1^{[i]}\mathbb{I}_3^T + \mathbb{I}_3\widetilde{P}_1^{[i]}\mathbb{I}_1^T + \mathbb{I}_1\left(\sum_{j=1}^r \sigma_j\left(\frac{d(t) - \tau_1}{\delta_\tau}(P_{mj} + X_1) + \frac{\tau_2 - d(t)}{\delta_\tau}(P_{1j} + X_2)\right)\right)\mathbb{I}_1^T,$$

$$\Psi_2^{[i]} = \widetilde{\Psi}^{[i]} - (\mathbb{I}_5 - \mathbb{I}_6)\frac{1}{\delta_\tau}\Lambda_{12}(\mathbb{I}_5 - \mathbb{I}_6)^T + \mathbb{I}_3\widetilde{R}\mathbb{I}_3^T + \mathbb{I}_1\widetilde{P}_2^{[i]}\mathbb{I}_3^T + \mathbb{I}_3\widetilde{P}_2^{[i]}\mathbb{I}_1^T + \mathbb{I}_1\left(\sum_{j=1}^r \sigma_j\left(\frac{d(t) - \tau_2}{\delta_\tau}(P_{2j} + X_3) + \frac{\tau_3 - d(t)}{\delta_\tau}(P_{mj} + X_4)\right)\right)\mathbb{I}_1^T,$$

$$\widetilde{\Psi}^{[i]} = diag\left\{\frac{1}{\delta_\tau}\dot{d}(t)(P_{mi} - P_{1i});\ -(1 - \dot{d}(t))Q;\ \left(\frac{\tau_1}{2}\right)^2(S_1 + S_2) + \delta_\tau^2(Z_1 + Z_2) + \delta_\tau R_3 + \tau_2 R_1;\ 0;\ Q;\ 0;\ 0\right\} + [\mathbb{I}_5\ \ \mathbb{I}_6]N[\mathbb{I}_5\ \ \mathbb{I}_6]^T$$
$$- [\mathbb{I}_6\ \ \mathbb{I}_7]N[\mathbb{I}_6\ \ \mathbb{I}_7]^T + [\mathbb{I}_1\ \ \mathbb{I}_4]M[\mathbb{I}_1\ \ \mathbb{I}_4]^T - [\mathbb{I}_4\ \ \mathbb{I}_5]M[\mathbb{I}_4\ \ \mathbb{I}_5]^T - (\mathbb{I}_1 - \mathbb{I}_4)(S_1 + U_1)(\mathbb{I}_1 - \mathbb{I}_4)^T - (\mathbb{I}_4 - \mathbb{I}_5)(S_2 + U_1)(\mathbb{I}_4 - \mathbb{I}_5)^T,$$

$$\Pi_\ell^{[i]} = \mathcal{F}_{1\ell}\left(A_i\mathbb{I}_1^T + A_{di}\mathbb{I}_2^T - \mathbb{I}_3^T\right) + \left(\mathbb{I}_1 A_i^T + \mathbb{I}_2 A_{di}^T - \mathbb{I}_3\right)\mathcal{F}_{1\ell}^T + \mathcal{F}_{2\ell}\Gamma_\ell^T + \Gamma_\ell\mathcal{F}_{2\ell}^T + \phi_\ell\alpha_{1i}^2\mathbb{I}_1 H_{1i}^T H_{1i}\mathbb{I}_1^T + \phi_\ell\alpha_{2i}^2\mathbb{I}_2 H_{2i}^T H_{2i}\mathbb{I}_2^T. \tag{15}$$

The matrices$\mathbb{I}_j$, $j = \{1, 2, ..., 7\}$, are block entry matrices with seven elements, e.g., $\mathbb{I}_4^T = [0\ \ 0\ \ 0\ \ I\ \ 0\ \ 0\ \ 0]$.

Proof First, we shall take the time derivative of (8) with respect to t along the trajectory of $x(t)$ which yields

$$\dot{V}_1(t) = \sum_{i=1}^{r}\Big[\rho_i(t)2\dot{x}^T(t)\Big(\chi\widetilde{P}_1^{[i]} + (1-\chi)\widetilde{P}_2^{[i]}\Big)x(t) + x^T(t)\dot{\rho}_i(t) \times\Big(\chi\widetilde{P}_1^{[i]} + (1-\chi)\widetilde{P}_2^{[i]}\Big)x(t) + \rho_i(t)\dot{d}(t)x^T(t)\frac{1}{\delta_\tau}(P_{mi} - P_{1i})x(t)\Big]$$
$$\dot{V}_2(t) = x^T(t-\tau_1)Qx(t-\tau_1) - (1-\dot{d}(t))x^T(t-d(t))Qx(t-d(t))$$
$$\dot{V}_3(t) = \begin{bmatrix}x(t-\tau_1)\\x(t-\tau_2)\end{bmatrix}^T N\begin{bmatrix}x(t-\tau_1)\\x(t-\tau_2)\end{bmatrix} - \begin{bmatrix}x(t-\tau_2)\\x(t-\tau_3)\end{bmatrix}^T N\begin{bmatrix}x(t-\tau_2)\\x(t-\tau_3)\end{bmatrix} + \begin{bmatrix}x(t)\\x(t-\frac{\tau_1}{2})\end{bmatrix}^T M\begin{bmatrix}x(t)\\x(t-\frac{\tau_1}{2})\end{bmatrix} - \begin{bmatrix}x(t-\frac{\tau_1}{2})\\x(t-\tau_1)\end{bmatrix}^T M\begin{bmatrix}x(t-\frac{\tau_1}{2})\\x(t-\tau_1)\end{bmatrix},$$
$$\dot{V}_4(t) = \dot{x}^T(t)\left((\frac{\tau_1}{2})^2(S_1+S_2) + \delta_\tau^2(Z_1+Z_2)\right) + \sum_{k=1}^{2}\frac{\tau_1}{2} \times\int_{t-\frac{k}{2}\tau_1}^{t-\frac{k-1}{2}\tau_1}\dot{x}^T(s)S_k\dot{x}(s)ds - \delta_\tau\int_{t-\tau_{k+1}}^{t-\tau_k}\dot{x}^T(s)Z_k\dot{x}(s)ds,$$
$$\dot{V}_5(t) = \dot{x}^T(t)(\tau_2R_1 + d(t)R_2 + \delta_\tau R_3 + (\tau_3-d(t))R_4)\dot{x}(t) - \int_{t-d(t)}^{t}\dot{x}^T(s)U_R\dot{x}(s)ds - \int_{t-\tau_3}^{t-d(t)}\dot{x}^T(s)(R_3+R_4)\dot{x}(s)ds, - \chi_{[\tau_1,\tau_2]}\int_{t-\tau_2}^{t-d(t)}\dot{x}^T(s)(R_1-R_3)\dot{x}(s)ds - (1-\chi_{[\tau_1,\tau_2]})\int_{t-d(t)}^{t-\tau_2}\dot{x}^T(s)(R_3-R_1)\dot{x}(s)ds, \tag{16}$$

where $\widetilde{P}_1^{[i]}$, $\widetilde{P}_2^{[i]}$, U_R are defined in (15). Considering (7), we introduce the symmetric matrices X_η, $\eta = \{1,2,3,4\}$,

$$\chi_{[\tau_1,\tau_2]}\sum_{i=1}^{r}\dot{\rho}_i(t)\left(\frac{d(t)-\tau_1}{\delta_\tau}X_1 + \frac{\tau_2-d(t)}{\delta_\tau}X_2\right) = 0,$$
$$(1-\chi_{[\tau_1,\tau_2]})\sum_{i=1}^{r}\dot{\rho}_i(t)\left(\frac{d(t)-\tau_2}{\delta_\tau}X_3 + \frac{\tau_3-d(t)}{\delta_\tau}X_4\right) = 0. \tag{17}$$

Combining (17) with the second term in $\dot{V}_1(t)$, and assuming that the conditions in Assumption 1 hold, we have

$$\dot{V}_1(t) = \sum_{i=1}^{r}\rho_i(t)2\dot{x}^T(t)\Big(\chi\widetilde{P}_1^{[i]} + (1-\chi)\widetilde{P}_2^{[i]}\Big)x(t) + \rho_i(t)\dot{d}(t)x^T(t) \times\frac{1}{\delta_\tau}(P_{mi}-P_{1i})x(t) + \sum_{j=1}^{r}\sigma_j x^T(t)\Big[\chi_{[\tau_1,\tau_2]}\Big(\frac{d(t)-\tau_1}{\delta_\tau}(P_{mj}+X_1) + \frac{\tau_2-d(t)}{\delta_\tau}(P_{1j}+X_2)\Big) + (1-\chi_{[\tau_1,\tau_2]})\Big(\frac{d(t)-\tau_2}{\delta_\tau}(P_{2j}+X_3) + \frac{\tau_3-d(t)}{\delta_\tau}(P_{mj}+X_4)\Big)x(t). \tag{18}$$

Moreover, considering free-weighting matrices, $\mathcal{F}_{11}$, $\mathcal{F}_{12}$, $\mathcal{F}_{21}$, $\mathcal{F}_{22}$, and denoting $\xi_{id}(t) := \frac{1}{d(t)-\tau_i}\int_{t-d(t)}^{t-\tau_i}\dot{x}(s)ds$, $i = \{1,2\}$ and $\xi_{dj}(t) := \frac{1}{\tau_j-d(t)}\int_{t-\tau_j}^{t-d(t)}\dot{x}(s)ds$, $j = \{2,3\}$, we introduce the following null expressions[1]

$$\chi_{[\tau_1,\tau_2]}2\zeta_x^T(\mathcal{F}_{21}(J_1(x(t-\tau_2)-x(t-d(t))+(\tau_2-d(t))\xi_{d2}) + J_2(x(t-d(t))-x(t-\tau_1)+(d(t)-\tau_1)\xi_{1d}))) = 0,$$
$$[1-\chi_{[\tau_1,\tau_2]}]2\zeta_x^T(\mathcal{F}_{22}(J_1(x(t-\tau_3)-x(t-d(t)) + (\tau_3-d(t))\xi_{d3}) + J_2(x(t-d(t))-x(t-\tau_2) + (d(t)-\tau_2)\xi_{2d}))) = 0,$$
$$2\zeta_x^T\left(\chi_{[\tau_1,\tau_2]}\mathcal{F}_{11}+[1-\chi_{[\tau_1,\tau_2]}]\mathcal{F}_{12}\right)\left(-\dot{x}(t)+(\overline{A}+\Delta\overline{A})x(t) + (\overline{A}_d+\Delta\overline{A}_d)x(t-d(t))+\overline{g}(t,x(t),x(t-d(t)))\right)=0, \tag{19}$$

where J_1, J_2 are defined in (15), and $\zeta_x^T(t) := [x^T(t)x^T(t-d(t))\ \dot{x}^T(t)\ x(t-\frac{\tau_1}{2})\ x(t-\tau_1)\ x(t-\tau_2)\ x(t-\tau_3)]$ Furthermore, applying the widely known Park-Moon's inequality,

$$2\zeta_x^T\mathcal{F}_{1\ell}\overline{g}(t) \le \phi_\ell^{-1}\zeta_x^T\mathcal{F}_{1\ell}\mathcal{F}_{1\ell}^T\zeta_x + \phi_\ell\overline{g}^T(t)\overline{g}(t), \quad \ell = \{1,2\}$$

with the quadratic constraint (3), we have

$$2\zeta_x^T(t)\mathcal{F}_{1\ell}\overline{g}(t,x(t),x(t-d(t))) \le \phi_\ell^{-1}\zeta_x^T\mathcal{F}_{1\ell}\mathcal{F}_{1\ell}^T\zeta_x + \phi_\ell x^T(t)\overline{H}_1x(t) + \phi_\ell x^T(t-d(t))\overline{H}_2x(t-d(t)), \tag{20}$$

for $\ell = \{1,2\}$, where $\overline{H}_1 = \sum_{i=1}^{r}\rho_i(t)\alpha_{1i}^2H_{1i}^TH_{1i}$, and $\overline{H}_2 = \sum_{i=1}^{r}\rho_i(t)\alpha_{2i}^2H_{2i}^TH_{2i}$.

At this point, we shall consider only the first subinterval $d(t)\in[\tau_1,\tau_2]$, i.e., $\chi_{[\tau_1,\tau_2]} = 1$. Expanding the integral terms and using Jensen's inequality (Lemma 1), yields

$$-\sum_{k=1}^{2}\int_{t-\frac{k}{2}\tau_1}^{t-\frac{k-1}{2}\tau_1}\dot{x}^T(s)\Big(\frac{\tau_1}{2}S_k + U_R\Big)\dot{x}(s)ds \le -\sum_{k=1}^{2}\int_{t-\frac{k}{2}\tau_1}^{t-\frac{k-1}{2}\tau_1}\dot{x}^T(s)ds(S_k+U_1)\int_{t-\frac{k}{2}\tau_1}^{t-\frac{k-1}{2}\tau_1}\dot{x}(s)ds,$$
$$-\int_{t-\tau_3}^{t-\tau_2}\delta_\tau\dot{x}^T(s)Z_2\dot{x}(s)ds \le -\int_{t-\tau_3}^{t-\tau_2}\dot{x}^T(s)dsZ_2\int_{t-\tau_3}^{t-\tau_2}\dot{x}(s)ds$$
$$-\int_{t-d(t)}^{t-\tau_1}\dot{x}^T(s)\Lambda_{12}\dot{x}(s)ds - \int_{t-\tau_2}^{t-d(t)}\dot{x}^T(s)\Lambda_{11}\dot{x}(s)ds \le -(d(t)-\tau_1)\xi_{1d}^T\Lambda_{12}\xi_{1d} - (\tau_2-d(t))\xi_{d2}^T\Lambda_{11}\xi_{d2}, \tag{21}$$

where Λ_{11}, $\Lambda12$ are defined in (15).

Now, combining the results from (16–21) and defining

$$\mho_\ell := \begin{bmatrix}\Psi_\ell + \overline{\Pi}_\ell & [(d(t)-\tau_\ell)\mathcal{F}_{2\ell}\ \ (\tau_{(\ell+1)}-d(t))\mathcal{F}_{2\ell}]\\ * & -diag\{(d(t)-\tau_\ell)\Lambda_{\ell 2};\ (\tau_{(\ell+1)}-d(t))\Lambda_{\ell 1}\}\end{bmatrix}, \tag{22}$$

where Ψ_1, Ψ_2, Λ_{11}, Λ_{12}, Λ_{21}, Λ_{22} are defined in (15), and $\overline{\Pi}_\ell = 2\mathcal{F}_{1\ell}(\overline{A}\mathbb{I}_1^T + \overline{A}_d\mathbb{I}_2^T - \mathbb{I}_3^T) + 2\mathcal{F}_{2\ell}\Gamma_\ell^T + \phi_\ell\mathbb{I}_1\overline{H}_1\mathbb{I}_1^T + \phi_\ell\mathbb{I}_2\overline{H}_2$
$\mathbb{I}_2^T + 2\mathcal{F}_{1\ell}(\Delta\overline{A}\mathbb{I}_1^T + \Delta\overline{A}_d\mathbb{I}_2^T) + \phi_\ell^{-1}\mathcal{F}_{1\ell}\mathcal{F}_{1\ell}^T$, we have the following LMI

$$\dot{V}(t)|_{d(t)<\tau_2} \le \zeta_1^T(t)(\mho_1)\zeta_1(t), \tag{23}$$

[1] As in (Figueredo et al. 2011), similar results may be achieved applying Finsler's Lemma (Oliveira and Skelton 2001).

with $\zeta_1^T(t) := [\,\zeta_x^T \;\; \xi_{1d}^T \;\; \xi_{d2}^T\,]$, and now, we shall consider the terms $\mho_{11}, \mho_{12}$ that arise from $\mho_1$ for $d(t) \to \tau_1$ and $d(t) \to \tau_2$, respectively. It is straightforward to conclude that the term on the right-hand side of the inequality (23) may be written as

$$\frac{\tau_2 - d(t)}{\tau_2 - \tau_1}\zeta_{11}^T(t)\mho_{11}\zeta_{11}(t) + \frac{d(t) - \tau_1}{\tau_2 - \tau_1}\zeta_{12}^T(t)\mho_{12}\zeta_{12}(t), \quad (24)$$

where $\zeta_{11}^T(t) := [\,\zeta_x^T \;\; \xi_{d2}^T\,]$ and $\zeta_{12}^T(t) := [\,\zeta_x^T \;\; \xi_{1d}^T\,]$. The analysis in (24) further enlightens the convex properties of $\mho_1$ regarding the time-varying delay (5). Therefore, it is easy to conclude that $\mho_1$ is negative definite if the vertices, $\mho_{11}$ and $\mho_{12}$, are also negative. Additionally, considering the properties of the time-varying matrix $\Delta(t)$, we employ the inequality $\left(2\overline{\alpha}_\ell^T \Delta(t)\overline{\beta} \le \epsilon_\ell \overline{\alpha}_\ell^T \overline{\alpha}_\ell + \epsilon_\ell^{-1}\overline{\beta}^T\overline{\beta}\right)$, $\ell = 1$, where $\overline{\alpha}_\ell = [\mathcal{F}_{1\ell}\overline{\Xi}_x \;\; 0]$, $\overline{\beta} = [(\overline{\Xi}_A \mathbb{I}_1^T + \overline{\Xi}_{Ad}\mathbb{I}_2^T) \;\; 0]$. Using Schur Lemma, it can be shown that the resulting LMIs are equivalent to $\sum_{i=1}^r \rho_i \Omega_{11}^{[i]} < 0$ and $\sum_{i=1}^r \rho_i \Omega_{12}^{[i]} < 0$, where $\Omega_{1k}^{[i]}$, $k = \{1, 2\}$ are defined in (13). Also, given (5b), we have that the matrices are convex in $\dot{d}(t) \in [d_{min}, d_{max}]$. Therefore, if the conditions in Theorem 1 are satisfied, then the inequality $\dot{V}(t)|_{d(t)<\tau_2} < 0$ holds.

Consider the case where $\tau_2 < d(t) \le \tau_3$, i.e., $\chi_{[\tau_1,\tau_2]} = 0$. Taking the results from (16–20), and applying Jensen's inequality (Lemma 1), similar to (21) but considering $\tau_2 < d(t) \le \tau_3$, we have

$$\dot{V}(t)|_{d(t)>\tau_2} \le \zeta_2^T(t)(\mho_2)\zeta_2(t),$$

where $\zeta_2^T(t) := [\,\zeta_x^T \;\; \xi_{2d}^T \;\; \xi_{d3}^T\,]$, and $\mho_2$ is defined in (22). Using analogous arguments from the former case, considering $\chi_{[\tau_1,\tau_2]} = 0$, we can conclude that $\mho_2$ is negative definite if $(\mho_2|_{d(t)\to\tau_2} < 0)$ and $(\mho_2|_{d(t)\to\tau_3} < 0)$ hold. Moreover, given (5b), we have that the matrices are convex in $\dot{d}(t) \in [d_{min}, d_{max}]$. Therefore, if the conditions in Theorem 1 hold, then the above conditions are satisfied and $\mho_2 < 0$.

We are now ready to complete the proof by establishing conditions that guarantee the negativeness of the Lyapunov functional's derivative. Note that the conditions in (12–13) implies $\sum_{i=1}^r \rho_i(t)\Omega_{\ell k}^{[i]} < 0$, $\ell, k = \{1, 2\}$, and, thus we must have $\dot{V}(t)|_{d(t)<\tau_2} < 0$ and $\dot{V}(t)|_{d(t)>\tau_2} < 0$. Furthermore, similar to (Fridman et al. 2009; Figueredo et al. 2010), it is easy to conclude that $\dot{V}(t)|_{d(t)=\tau_2} < 0$ holds. Therefore, if the conditions in Theorem 1 are satisfied, then the T–S fuzzy time-delay system with local uncertain nonlinear systems is robustly asymptotically stable, which concludes the proof. □

Remark 1 It is also important to consider two particular cases regarding the information on the delay and it's derivative: the case when the time-varying delay derivative lower bound is unknown, and the case when there exists no information concerning the time-delay derivative, i.e., fast-varying delays. Theorem 1 can be easily adapted to deal with both cases. For the former, when the lower bound for the delay derivative d_{min} is unknown, if we consider

$$R_2 > R_4, \quad P_{2i} > P_{1i}, \quad i = \{1, ..., r\}, \quad (25)$$

instead of the conditions (12a) and (13a), the results from Theorem 1 shall be valid for unknown d_{min}. Note that, if (25) holds, then the conditions (12b) and (13b), if satisfied, yield (12a) and (13a). An evident consequence is the needlessness of the derivative lower bound information for the resulting stability conditions. For the later case, i.e., when no restrictions are cast upon the derivative, assuming $P_{mi} = P_{1i} = P_{2i}$, and null Q, R_2, R_4 matrices, all the information concerning the delay derivative is removed from Theorem 1 conditions, and thus one may apply the results for fast-varying delays.

Remark 2 The fuzzy weighting-dependent Lyapunov-Krasovskii functional as constructed in (8) yields conditions dependent on the membership functions time-derivative. By choosing constant matrices $P_{1i} = P_1 > 0$ and $P_{2i} = P_2 > 0$, $i = \{1, \ldots, r\}$, (8) loses the fuzzy weighting-dependence and is regarded as a quadratic Lyapunov-Krasovskii functional which does not depend on Assumption 1. The conditions in Theorem 1 thus become independent of the knowledge about the membership function.

Theorem 1 provides stability conditions for a new class of T–S fuzzy models with local uncertain nonlinear time-delay systems, and is the main result of the paper. The proposed stability analysis introduced a novel fuzzy weighting-dependent Lyapunov-Krasovskii functional which incorporates state-of-the-art stability techniques for linear time-delay systems with an improved piecewise analysis method, amended with new delay-interval-dependent and fuzzy weighting-dependent LKF terms. Compared to ordinary PAM-based criteria, by deeper exploiting the PAM's delay-intervals, we have weakened the positiveness restrictions upon these new terms, whereas maintaining the LKF positive. In this context, the method further exploited the delay derivative information, and relaxed conditions upon resulting LMIs, yielding less conservative results.

Further improvements may be obtained by increasing the number of delay range partitions. Nevertheless, the improvements are relatively small in face to the increment on the number of variables and LMIs conditions, which in turn increases substantially the solution complexity. Therefore, to increase the number of partitions is only recommended for particular cases with very large delay ranges ($\tau_{min} \to 0$ and $\tau_{max} \to \infty$).

Remark 3 The solution complexity from Theorem 1 depends on the dimension of the system r_x, the number of fuzzy rules r, and on the functions $k_1 = k_1(d(t))$, $k_2 = k_2(x(t))$, and $k_3 = k_3(V(t))$, where the latter are indicator functions which are assumed to be 1, if there exists information regarding the delay derivative (5b) and the model uncertainties (4), and if Assumption 1 holds, respectively. In this context, the number of scalar variables K, apart from free-weighting matrices, and LMI rows L used in the proposed criterion are calculated as follow

$$K = 4k_2(t) + (9 + 2k_1(t) + (2 + r + (2 + r)k_1(t))k_3(t))r_x \times \frac{r_x + 1}{2};$$
$$L = 4r_x + r(4 + 4k_1(t))(8 + 3k_2(t))r_x + (7 + 2rk_3(t))r_x.$$

The complexity from the criterion is slightly higher than traditional techniques, for the piecewise analysis technique combined with the convex optimization yields additional scalar variables from slack matrices (in our case, $42r_x^2$), see (Fridman et al. 2009; Orihuela et al. 2010; Figueredo et al. 2010).

4 Numerical Examples

In this section, different benchmark examples are exploited to illustrate the effectiveness of the proposed criteria. First, we present three examples that highlight the advantages and the conservatism reduction of our method compared to state-of-the-art criteria for conventional T–S fuzzy time-delay systems. In the fourth example, we investigate the possibility and the improvements of applying Theorem 1 to the stability analysis of linear time-delay systems. Finally, we illustrate the effectiveness of the proposed method for the analysis of T–S fuzzy models with local nonlinear time-delay systems.

Example 1 Consider the nominal T–S fuzzy time-delay system

$$\dot{x}(t) = \sum_{i=1}^{2} \rho_i(\theta(t)) \left[A_i x(t) + A_{di} x(t - d(t))\right] \tag{26}$$

with $\rho_1(\theta(t)) = (1 + exp(-2x_1(t)))^{-1}, \rho_2(\theta(t)) = (1 - \rho_1)$, and

$$A_1 = \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}.$$

Assuming $\tau_{min} = 0$ and unknown derivative lower bound, d_{min}, the maximum τ_{max} which maintain the T–S fuzzy system asymptotic stability for various d_{max} are listed in Table 1. The Table 1 explicitly illustrates that the proposed stability criterion when applied to conventional T–S fuzzy time-delay systems yields considerably superior results compared to previous criteria in the literature. Particularly for fast-varying delay, the improvements over results from (Liu et al. 2010) are as high as 115 %.

Table 1 also shows the dependence relationship between the maximum allowed delay and the maximum delay growing velocity (upper bound of the delay derivative). As it would be expected, τ_{max} grows as $d_{max} \to 0$. This is highlighted in the difference between the results from Theorem 1 for $d_{max} = 0$ and for fast-varying delays. Note that further improvements may be obtained when d_{min} is known and $d_{min} \to 0$, e.g., for fixed $d_{max} = 0.5$ and $\tau_{min} = 0$, we have $\tau_{max} = 1.95$ and 2.01 for $d_{min} = -1.0$ and -0.1, respectively.

Table 1 Admissible upper bound values of τ_{max} for unknown d_{min} and $\tau_{min} = 0$ (Example 1)

Methods $\diagdown d_{max}$	0	0.1	0.5	Unknown
Guan and Chen (2004)	1.25	–	–	–
Chen et al. (2007)	3.15	–	–	–
Liu et al. (2010)	3.30	2.65	1.50	0.79
Theorem 1	4.12	3.22	1.92	1.70

Table 2 Max. τ_{max} for fast-varying delays, $\tau_{min} = 0$, and various values for the bounding parameter σ (Example 1)

max σ :	0.1	0.2	0.5	1.0	2.0	unknown
τ_{max} :	2.590	2.547	2.446	2.380	2.372	2.370

Moreover, suppose we replace the above membership functions by ones satisfying (7) and Assumption 1 with $\dot{\rho}_i(\theta(t)) \le \sigma$, $i = \{1, 2\}$.

For fast-varying delay with $\tau_{min} = 0$, and different values of σ, the results from Theorem 1 are shown in Table 2. It is clear that τ_{max} grows as we further decrease the bounds for σ, i.e., $\sigma \to 0$. This analysis reveals the benefits of considering the fuzzy weighting-dependent LKF.

Example 2 Consider the T–S fuzzy time-delay system

$$\dot{x}(t) = \sum_{i=1}^{2} \rho_i(\theta(t)) \left[A_i x(t) + A_{di} x(t - d(t))\right] \tag{27}$$

with

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}, \tag{28}$$

and membership functions $\rho_i(\theta(t))$ satisfying (7) and Assumption 1 with $\dot{\rho}_i(\theta(t)) \le \sigma_i = 0.5$, $i = \{1, 2\}$. For fast-varying delay and $\tau_{min} = 0.4$, the maximum delay upper bound from Theorem 1 is 1.460, whereas the state-of-the-art technique for time-delay fuzzy systems regarding Assumption 1, (Zhao et al. 2011), yields 1.380.

Now, in order to allow fair comparison with different criteria, throughout this example we shall only consider the results from Theorem 1 disregarding Assumption 1. In this context, for fast-varying delay and various values of τ_{min}, the results from applying Theorem 1, with (8) regarded as a quadratic LKF, are presented in Table 3. The results enlighten the importance and advantages of the proposed method compared to state-of-the-art criteria for conventional T–S fuzzy time-delay systems. The admissible τ_{max} from Theorem 1 is up to 25 % superior (for $\tau_{min} = 0$) compared to the state-of-the-art criterion, (Peng and Han 2011).

Example 3 Suppose a T–S fuzzy time-delay system with model uncertainties

$$\dot{x}(t)=\sum_{i=1}^{2}\rho_i[(A_i+\Delta A_i)x(t)+(A_{di}+\Delta A_{di})x(t-d(t))],$$

where A_i, A_{di}, $i = \{1, 2\}$ are defined in (28), and $\Xi_{xi} = I$, $\Xi_{Ai} = diag\{0.1; 0.1\}$, and $\Xi_{Adi} = diag\{0.1; 0.1\}$, for $i = \{1, 2\}$. In order to compare with the previous results, we set the system time-delay to be invariant, i.e., with zero derivative ($\dot{d}(t) = 0$). In this particular case, the maximum τ_{max} value obtained from (Chen et al. 2007) and (Peng et al. 2009b) are, respectively, 1.352, 1.447, whereas the result from Theorem 1 is 1.570.

Example 4 Consider the following linear time-delay system with no uncertainties

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-d(t)),$$

For various values of τ_{min} and time-delay derivative varying within $\dot{d}(t) \in [-0.3,\ 0.3]$, the allowable τ_{max} values that maintain the asymptotic stability are presented in Table 4.

The method proposed in Theorem 1 when particularized to linear time-delay systems, i.e., system (6) with only one fuzzy rule and $g_1(t, x(t), x(t-d(t))) = 0$, yields slightly superior results compared to the state-of-the-art criterion in linear time-delay systems literature (Figueredo et al. 2011). Moreover, compared with the results from different authors (Shao 2009; Fridman et al. 2009; Sun et al. 2010), the admissible values for τ_{max} from Theorem 1 are considerably less conservative.

Example 5 We shall now consider a nonlinear time-delay system that can be described using conventional T–S fuzzy model, as in Example 2, but subjected to an additional highly complex nonlinearity, e.g.,

$$\vartheta(t, x(t)) = \beta sin(x_1(t)) cos^3(x_2(t)) sgn(\psi(x)) \sqrt{|\psi(x)|}, \quad (29)$$

where $\psi(x(t)) = x_1^2(t) + 2x_1(t)x_2(t) + \frac{1}{2}x_2^2(t)$, and β is a known bounding constant. If we take the new class of T–S fuzzy models (6) with local time-delay systems subjected to nonlinearities, the system can thus be easily described by

$$\dot{x}(t) = \sum_{i=1}^{2}\rho_i[A_i x(t) + A_{di}x(t-d(t)) + g_i(t, x(t))]$$

with similar membership functions and matrices, A_i and A_{di},

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix};$$

and $g_i(t, x(t), x(t-d(t))) = \vartheta(t, x(t))$, $i = \{1, 2\}$. It can be shown that the nonlinearities satisfy the quadratic constraint (3) with $H_i = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $\alpha_{1i} = \beta, \alpha_{2i} = 0$.

Table 3 Max. τ_{max} for fast-varying delays (Example 2)

Methods $\diagdown \tau_{min}$	0	0.4	0.8	1.0	1.2
Tian and Peng (2006)	0.721	0.884	1.094	1.211	1.337
Lien et al. (2007)	0.831	0.883	1.068	–	1.318
Li et al. (2009)	0.982	1.038	1.158	1.252	1.359
Peng et al. (2009a)	0.982	1.182	1.313	–	1.433
Peng and Han (2011)	1.078	1.162	1.281	–	1.429
Theorem 1	1.338	1.338	1.376	1.433	1.512

Note that '– 'means the authors did not provide results for prescribed conditions

Table 4 Allowable τ_{max} value for $|\dot{d}(t)| \leq 0.3$ (Example 4)

Methods $\diagdown \tau_{min}$	0	1	2	3	4
Shao (2009)	–	–	2.697	3.259	4.074
Sun et al. (2010)	–	–	3.013	3.341	4.169
Fridman et al. (2009)					
Thm 1	3.052	3.179	2.961	3.321	4.090
Thm 2	2.811	3.114	3.153	3.458	4.257
Figueredo et al. (2010)	3.052	3.185	3.190	3.464	4.257
Figueredo et al. (2011)	3.064	3.190	3.198	3.480	4.257
Theorem 1	3.065	3.191	3.203	3.488	4.260

Table 5 Allowable values of τ_{max} for fast-varying delay and various values for the bounding parameter, β, (Example 5)

	$\beta^2 = 0.1$	$\beta^2 = 0.2$	$\beta^2 = 0.3$	$\beta^2 = 0.5$	$\beta^2 = 1.0$
$\tau_{min} = 0$	1.170	1.088	1.018	0.908	0.667
$\tau_{min} = 0.4$	1.161	1.067	0.995	0.882	0.525
$\tau_{min} = 0.8$	1.159	1.082	1.030	0.945	–

Assuming a scenario with fast-varying delay, our purpose is to find the maximum value of τ_{max} that maintains the asymptotic stability of the new class of T–S fuzzy system considering different values for the bounding parameter, β. The results are listed in Table 5, and explicit the effectiveness of the proposed analysis technique.

5 Conclusion

We developed a robust stability theorem for nonlinear time-delay systems described by a new class of T–S fuzzy models with local nonlinear systems subjected to model uncertainties and time-varying delay. The new fuzzy model, for being obviously more general than conventional T–S fuzzy models with local linear systems, eases the T–S fuzzy description whereas reducing the number of fuzzy rules, and, therefore, is more suitable for practical implementations. Our stability results were presented as LMIs and considered the delay derivative upper and lower bounded. As special cases, we also considered unknown derivative lower bound, and when no restrictions are cast upon the derivative. To reduce conservatism concerning both the novel and the conventional T–S fuzzy time-delay systems, we developed a new fuzzy weighting-dependent Lyapunov-Krasovskii functional combining state-of-the-art stability techniques for linear time-delay systems with an improved piecewise analysis method, which makes use of novel and less-restricted delay-interval-dependent LKF terms. It should be likewise acknowledged that the results particularized for the stability analysis of linear time-delay systems are also superior compared to previous criteria in time-delay literature. The advantages of the proposed method were further enlightened with numerical examples that illustrate the significant conservativeness reduction compared to state-of-the-art criteria, and the analysis effectiveness for the new class of T–S fuzzy models with local nonlinear time-delay systems.

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