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Robust Expansion of Uncertain Volterra Kernels into Orthonormal Series

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Abstract— **This paper is concerned with the computation of uncertainty bounds for the expansion of uncertain Volterra models into an orthonormal basis of functions, such as the Laguerre or Kautz bases. This problem has already been addressed in the context of linear systems by means of an approach in which the uncertainty bounds of the expansion coefficients have been estimated from a structured set of impulse responses describing a linear uncertain process. This approach is extended here towards nonlinear Volterra models through the computation of the uncertainty bounds of the expansion coefficients from a structured set of uncertain Volterra kernels. The proposed formulation assures that the resulting model is able to represent all the original uncertainties with minimum intervals for the expansion coefficients. An example is presented to illustrate the effectiveness of the proposed formulation.**

I. Introduction

The performance of a controlled closed loop system depends on the reliability of the model used for the controller synthesis. In certain cases, however, a single model cannot adequately represent a complex system. The presence of external disturbances, for example, may rule out obtaining a single set of model parameters that would lead to a good representation of the system. A usual procedure to deal with this sort of situation consists in incorporating into the model uncertainties associated with its parameters [15], [7], [28], [20]. When the uncertainties on the model parameters are described by means of intervals and the model order is known, it is said that the model uncertainty is of structured type [7], [10]. Structured uncertainties are often constrained to geometric regions of the parameters space, such as polytopes, orthotopes, or ellipsoids [28], [9]. Much research has focused on the modeling of dynamic systems with structured uncertainties. Models with structured uncertainties are the basis of the so-called robust control algorithms (e.g. see [14], [30]).

In the past few decades, there has been a growing interest in the use of orthonormal basis functions (OBF), such as the Laguerre and Kautz functions, in studies involving the identification and control of dynamic processes [12], [3], [20], [21], [13]. The main reason for using OBF in such areas is that the corresponding approximate (modeling and control) problems usually have simpler solutions, as the orthonormality of these functions often yields simpler general models. An important issue regarding the use of an orthonormal basis model structure is the incorporation of approximate knowledge about the dynamics of the system into the identification process [6], [18]. This allows a significant reduction of the number of model parameters to be estimated, thus reducing the variance of their estimates [19]. The consequence is an increase in the accuracy of the results.

A few approaches can be found in the literature for the estimation of uncertain parameters in OBF-based models, both in the context of linear [25], [1] and nonlinear (e.g. Wiener and Hammerstein) systems [11], [2]. One such approach [22] involves a set of I/O data measured from the system and the use of robust identification methods based on the unknown-but-bounded-error (UBBE) strategy [15]. In this case, the uncertainty bounds can be arbitrarily chosen, what involves the following two risks: *i)* On the one hand, if the bounds are under-estimated, the problem of the robust identification of the parameters may have no solution; *ii)* On the other hand, if the bounds are over-estimated, then the solution can be very conservative. A different approach considers prior knowledge of structured uncertainties associated with the impulse response of a linear process [22]. However, as it will be shown by means of an example in Section V, this approach cannot always guarantee that the resulting OBF model represents all the original uncertainties of the process. A solution to this problem in the context of linear models has been described in [16] as a solution to a constrained optimization problem. The formulation of such an optimization problem and the corresponding solution are extended in the present paper to the context of nonlinear Volterra models.

The outline of this paper is as follows. In the next section, OBF-based models are briefly reviewed. In Section III, the formulation of uncertain models using interval-valued coefficients is presented. In Section IV, optimization problems for computing the bounds for the expansion coefficients of uncertain Volterra kernels are proposed. In Section V, an example illustrating the theoretical results is presented and, finally, Section VI addresses the conclusions.

II. Approximation of Volterra models using orthonormal functions

A Volterra model is essentially an input-output functional (polynomial) expansion of a nonlinear system whose structure is given by a straightforward generalization of

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the unit-impulse response model [8], [17]. In the discrete-time domain, the mathematical description of an ηth-order Volterra model relates the output $y(k)$ of a physical process to its input $u(k)$ as [23], [17]:

$$
\begin{aligned}
y(k) = &\sum_{\tau_1=0}^{\infty} h_1(\tau_1)u(k-\tau_1) + \\
&\sum_{\tau_1=0}^{\infty}\sum_{\tau_2=0}^{\infty} h_2(\tau_1,\tau_2)u(k-\tau_1)u(k-\tau_2) \ + \ \cdots \\
&+ \sum_{\tau_1=0}^{\infty}\cdots\sum_{\tau_\eta=0}^{\infty} h_\eta(\tau_1,\ldots,\tau_\eta)u(k-\tau_1)\cdots u(k-\tau_\eta), \quad (1)
\end{aligned}
$$

where the multidimensional functions $h_\eta(\tau_1,\ldots,\tau_\eta)$ are the ηth-order Volterra kernels. Although these models can describe a wide class of nonlinear systems [4], [23], their practical use is limited due to the usually large number of coefficients to be estimated. Such a drawback can be avoided by expanding the Volterra kernels using OBF. The number of parameters necessary to represent the models can thus be drastically reduced if properly designed bases of functions are adopted.

Representing a nonlinear dynamic system by means of an orthonormal series expansion gives rise to a model of Wiener type [29]. This sort of model consists of a linear dynamic, here composed of a set of orthonormal filters, followed by a nonlinear static mapping, here represented by the Volterra series. The basic idea of such OBF-Volterra models is to mathematically describe the Volterra kernels h_η with an orthonormal basis of functions $\{\psi_m\}$, as [23], [17]:

$$
h_\eta(k_1,\ldots,k_\eta) = \sum_{i_1=1}^{\infty}\cdots\sum_{i_\eta=1}^{\infty}\alpha_{i_1,\ldots,i_\eta}\prod_{l=1}^{\eta}\psi_{\eta,i_l}(k_l), \quad (2)
$$

which assumes that the kernels are absolutely summable on $[0,\infty)$. In practice, this condition can be assured if the long memory terms of the kernels are null, which is possible provided that the system to be modeled is stable. In other words, $h_\eta(k_1,\ldots,k_\eta)$ is assumed to be zero for $k_l > \epsilon, \ \ \forall \ l \in \{1,\ldots,\eta\}$. An appropriate value for $\epsilon < \infty$ can be set based on the settling time of the system.

The kernel expansion coefficients $\alpha_{(\cdot)}$ in (2) can be derived using the orthonormality property of the set $\{\psi_m\}$, i.e., $\sum_{k=0}^{\infty}\psi_q(k)\psi_r(k) = \delta_{qr}$, where δ_{qr} is the Kronecker delta, as

$$
\alpha_{i_1,\ldots,i_\eta} = \sum_{k_1=0}^{\infty}\cdots\sum_{k_\eta=0}^{\infty} h_\eta(k_1,\ldots,k_\eta)\prod_{l=1}^{\eta}\psi_{\eta,i_l}(k_l). \quad (3)
$$

For computational reasons, equation (2) is, in practice, approximated with a finite number M of functions, as follows:

$$
\hat{h}_\eta(k_1,\ldots,k_\eta) = \sum_{i_1=1}^{M}\cdots\sum_{i_\eta=1}^{M}\alpha_{i_1,\ldots,i_\eta}\prod_{l=1}^{\eta}\psi_{\eta,i_l}(k_l). \quad (4)
$$

The use of orthonormal functions to represent signals and systems has a long history, since the pioneering works in [24] and [29]. Discrete-time orthonormal basis functions can be generated by cascading different all-pass filters of order one or two, as follows [13], [18]:

$$
\Psi_m(z) \ = \ \frac{\sqrt{1-|\beta_m|^2}}{z-\beta_m}\prod_{j=1}^{m-1}\left(\frac{1-\bar{\beta}_j z}{z-\beta_j}\right) \quad (5)
$$
$$
m = 1,2,\ldots,
$$

where $\beta_m, \bar{\beta}_m$ are the stable poles of the orthonormal basis ($\beta_m \in \mathbb{C} : |\beta_m| < 1$). The corresponding realizations in time-domain, $\psi_m(k)$, are given by the inverse Z-transform of (5) and satisfy the orthonormality property. The set $\{\psi_m\}$ is complete on $\ell^2[0,\infty)$ if and only if $\sum_{m=1}^{\infty}(1-|\beta_m|) = \infty$ [18], [13], so any finite energy signal (including absolutely summable kernels) can be approximated with any prescribed accuracy by linearly combining a certain finite number of such functions.

When all the poles of (5) are real-valued and equal to each other, i.e., $\beta_m = \bar{\beta}_m = c$, one gets the Laguerre basis, which can be written in the z-domain as [27], [13]:

$$
\Psi_m(z) = \frac{\sqrt{1-c^2}}{z-c}\left(\frac{1-cz}{z-c}\right)^{m-1}, \quad (6)
$$

with c denoting the Laguerre pole.

Another important OBF realization, which has also been shown to be a particular case of a unifying definition for (5) [18], is obtained by cascading an all-pass filter with pole at β and an all-pass filter with pole at $\bar{\beta}$, where $\bar{\beta}$ denotes the complex conjugate of β. By setting the pairs of conjugate poles equal to each other for any value of m, i.e., $\{\beta,\bar{\beta},\beta,\bar{\beta},\ldots\}$, the result is the so-called two-parameter Kautz functions. These functions are defined in the z-domain as [26], [13]:

$$
\Psi_{2m}(z) = \frac{\sqrt{(1-c^2)(1-b^2)}}{z^2+b(c-1)z-c} \cdot \left[\frac{-cz^2+b(c-1)z+1}{z^2+b(c-1)z-c}\right]^{m-1},
$$

$$
\Psi_{2m-1}(z) = \frac{(z-b)\sqrt{1-c^2}}{z^2+b(c-1)z-c} \cdot \left[\frac{-cz^2+b(c-1)z+1}{z^2+b(c-1)z-c}\right]^{m-1}, \quad (7)
$$

where the real-valued parameters b and c are related to the pair of Kautz poles $(\beta,\bar{\beta})$ as $b = (\beta+\bar{\beta})/(1+\beta\bar{\beta})$ and $c = -\beta\bar{\beta}$.

III. Structured Uncertainties in OBF-Volterra Models

The robust identification method to be studied here takes as input a set of realizations of the Volterra kernels. Such a set represents the process uncertainties. So, let us consider an ηth-order Volterra model, as that in (1), which has uncertainties on its parameters. Such uncertainties can be mathematically described by the following set of kernels:

$$\{h_\eta(k_1,\ldots,k_\eta)\} = \Big\{\bar{h}_\eta(k_1,\ldots,k_\eta) + \rho_\eta(k_1,\ldots,k_\eta)\Delta h_\eta(k_1,\ldots,k_\eta)\Big\}, \quad (8)$$

where $\bar{h}_\eta(k_1,\ldots,k_\eta)$ is the mean ηth-order Volterra kernel, $\Delta h_\eta(k_1,\ldots,k_\eta) \geq 0$ is the maximum deviation from the mean at each time instant, and $\rho_\eta(k_1,\ldots,k_\eta)$ is such that $|\rho_\eta(k_1,\ldots,k_\eta)| \leq 1$. Using (8) to rewrite the expansion coefficients (3), one gets a set of uncertain coefficients, as follows:

$$\{\alpha_{i_1,\ldots,i_\eta}\} = \sum_{k_1=0}^{\infty}\cdots\sum_{k_\eta=0}^{\infty}\Big\{\bar{h}_\eta(k_1,\ldots,k_\eta) + \rho_\eta(k_1,\ldots,k_\eta)\Delta h_\eta(k_1,\ldots,k_\eta)\Big\}\prod_{l=1}^{\eta}\psi_{\eta,i_l}(k_l). \quad (9)$$

Their nominal values are described in terms of the mean Volterra kernels $\bar{h}_\eta(k_1,\ldots,k_\eta)$, whereas the deviation values are described in terms of Δh_η, as follows:

$$\bar{\alpha}_{i_1,\ldots,i_\eta} = \sum_{k_1=0}^{\infty}\cdots\sum_{k_\eta=0}^{\infty}\bar{h}_\eta(k_1,\ldots,k_\eta)\prod_{l=1}^{\eta}\psi_{\eta,i_l}(k_l), \quad (10)$$

$$\Delta\alpha_{i_1,\ldots,i_\eta} = \sum_{k_1=0}^{\infty}\cdots\sum_{k_\eta=0}^{\infty}\Delta h_\eta(k_1,\ldots,k_\eta)\prod_{l=1}^{\eta}\psi_{\eta,i_l}(k_l). \quad (11)$$

From the development presented above, the set of coefficients can be rewritten as:

$$\{\alpha_{i_1,\ldots,i_\eta}\} = \{\bar{\alpha}_{i_1,\ldots,i_\eta} + \sigma_{i_1,\ldots,i_\eta}\Delta\alpha_{i_1,\ldots,i_\eta}\}, \quad (12)$$

where $\sigma_{i_1,\ldots,i_\eta}$ are scaling terms. Unlike $\bar{\alpha}_{i_1,\ldots,i_\eta}$ and $\Delta\alpha_{i_1,\ldots,i_\eta}$, these scaling terms cannot be analytically derived from (9). The reason is that, unlike $\bar{h}_\eta(k_1,\ldots,k_\eta)$ and Δh_η, which are unique, the terms $\rho_\eta(k_1,\ldots,k_\eta)$ in (9) are not unique for the whole set of coefficients $\{\alpha_{i_1,\ldots,i_\eta}\}$. Hence, the scaling terms $\sigma_{i_1,\ldots,i_\eta}$ must be numerically estimated. An optimal formulation to estimate these scaling terms is described in the next section. Such a formulation is aimed at minimizing the uncertainties associated with the expansion coefficients, i.e., the second term of the right-hand side of expression (12).

IV. Computation of Uncertainty Bounds

This section introduces optimization problems for computing uncertainty bounds for uncertain dynamical systems represented by OBF models. First, a formulation for linear models previously introduced in the literature is reviewed. Next, such a formulation is extended to any-order Volterra models. The original and extended formulations are valid regardless of the orthonormal basis adopted (Laguerre, Kautz, or Generalized Orthonormal Basis Functions — GOBF).

A. Linear models

An optimization problem whose solution provides an overestimate of the set of impulse responses of uncertain linear systems has been proposed in [16]. Provided that set $\{h_1(k_1)\}$ is known in advance, the formulation of the problem is as follows:

$$\begin{aligned}
&\min_{\sigma_{i_1}} \prod_{i_1=1}^{M}\sigma_{i_1}\Delta\alpha_{i_1},\\
&\text{s.t. } \{h_1(k)\} \subseteq \{\breve{h}_1(k)\},\\
&\qquad \breve{h}_1(k) = \sum_{i_1=1}^{M}\big(\bar{\alpha}_{i_1} + \sigma_{i_1}\Delta\alpha_{i_1}\big)\psi_{i_1}(k),
\end{aligned} \quad (13)$$

where

$$\bar{\alpha}_{i_1} = \sum_{k=0}^{\infty}\bar{h}_1(k)\psi_{i_1}(k), \quad (14)$$

$$\Delta\alpha_{i_1} = \sum_{k=0}^{\infty}\Delta h_1(k)\psi_{i_1}(k), \quad (15)$$

for $i_1 = 1,\ldots,M$.

The objective function of problem (13) describes the volume of the hyper-rectangle generated by varying the uncertainties associated with the orthonormal expansion coefficients, i.e., $\sigma_{i_1}\Delta\alpha_{i_1}$. The deviations of these coefficients, $\Delta\alpha_{i_1}$ $(i_1 = 1,\ldots,M)$, are scaled by σ_{i_1}. The problem consists in minimizing the volume of the hyper-rectangle that bounds the uncertainties, subject to the constraint that the system impulse responses, represented by the set $\{h_1(k)\}$, must lie within the set of model impulse responses $\{\breve{h}_1(k)\}$. The variables of the optimization problem (13) are the coefficients σ_{i_1} $(i_1 = 1,\ldots,M)$. The constraints of (13) assure that the resulting model represents all the impulse responses of the process.

B. Nonlinear models

Consider an ηth-order Volterra model with uncertain kernels, just like those in (8). According to (12), the uncertainties over the corresponding OBF expansion coefficients are represented by the terms $\sigma_{i_1}\Delta\alpha_{i_1}$, $\sigma_{i_1,i_2}\Delta\alpha_{i_1,i_2}$, and $\sigma_{i_1,\ldots,i_\eta}\Delta\alpha_{i_1,\ldots,i_\eta}$ for the 1st, 2nd, and ηth orders of the model, respectively. Since the Volterra kernels are independent from each other, that is, there is no mutual dependence among $h_1, h_2, \ldots, h_\eta$, then it is proposed here that η independent optimization problems be formulated as in (16), where $\bar{\alpha}_{i_1,\ldots,i_\eta}$ and $\Delta\alpha_{i_1,\ldots,i_\eta}$ are given by (10) and (11), respectively. As in Section IV-A, it is assumed here that the set of kernels $\{h_\eta(k_1,\ldots,k_\eta)\}$ is known in advance.

Analogously to (13), the objective in (16) is to minimize the volumes generated by the scaling of the uncertainties over the expansion coefficients $\alpha_{(\cdot)}$. In an ηth-order Volterra model, the optimal solutions of the η independent problems minimize the volumes generated by the corresponding uncertainties, subject to the constraint that every possible kernel of the original Volterra model must lie within the set of OBF model kernels, $\{\breve{h}_l(k_1,\ldots,k_l)\}$.

$$\min_{\sigma_{i_1,\ldots,i_\eta}} \prod_{i_1=1}^{M} \cdots \prod_{i_\eta=1}^{M} \sigma_{i_1,\ldots,i_\eta} \Delta\alpha_{i_1,\ldots,i_\eta},$$
$$\text{s.t. } \{h_\eta(k_1,\ldots,k_\eta)\} \subseteq \{\breve{h}_\eta(k_1,\ldots,k_\eta)\}, \quad (16)$$
$$\breve{h}_\eta(k_1,\ldots,k_\eta) = \sum_{i_1=1}^{M} \cdots \sum_{i_\eta=1}^{M} \left(\bar{\alpha}_{i_1,\ldots,i_\eta} + \sigma_{i_1,\ldots,i_\eta}\Delta\alpha_{i_1,\ldots,i_\eta}\right) \prod_{l=1}^{\eta} \psi_{\eta,i_l}(k_l),$$
$$i_l = 1,\ldots,M.$$

The variables of the optimization problems in (16) are the coefficients $\sigma_{i_1,\ldots,i_l}$ for $l = 1,\ldots,\eta$. Their optimal values provide the minimal scaling of the sides of the hyper-rectangle that represents the OBF model uncertainties so that any element of the sets of original kernels is comprised by the resulting model.

Problem (16) can be solved by using a standard constrained nonlinear optimization method. For instance, the MATLAB routine `fmincon`, which performs a numerical search for a constrained minimum of a function of several variables, can be used.

V. Example

The goal of this example is to obtain uncertainty bounds in OBF-Volterra models using the method proposed in Section IV-B. Consider a second-order Volterra model with the following uncertain kernels:

$$h_1(k_1) = \mathcal{Z}^{-1}\left[\frac{z+0.5}{z^2-1.2z+r_1}\right], \quad (17)$$
$$h_2(k_1,k_2) = g_{i_1}(k_1)g_{i_2}(k_2), \quad (18)$$

with

$$g_{i_l}(k) = \mathcal{Z}^{-1}\left[\frac{z+1}{z^2-r_{i_l}z+0.5}\right], \quad (19)$$

where $\mathcal{Z}^{-1}$ denotes the unilateral inverse Z-transform. The kernel uncertainties are given in terms of uncertain parameters of the model, here represented by r_1 and (r_{i_1}, r_{i_2}), respectively. Suppose that such parameters lie on intervals: $r_1 \in [0.3, 0.5]$ and $(r_{i_1}, r_{i_2}) \in [0.4, 0.8] \times [0.4, 0.8]$. For computational reasons, these intervals were sampled using a sampling increment of 0.01.

The central first- and second-order kernels, i.e., $\bar{h}_1(k_1)$ and $\bar{h}_2(k_1,k_2)$, are the mean values of $h_1(k_1)$ and $h_2(k_1,k_2)$, respectively, at each time instant. In other words, $\bar{h}_1$ is the average of all the possible functions in (17), by varying the parameter r_1 within its feasibility interval, as in [16]. The same reasoning is valid for $\bar{h}_2$ and (r_{i_1}, r_{i_2}) in (19). Mathematically, one has the following:

$$\begin{aligned} r_1(l) &= 0.3 + l\cdot 0.01, \quad l = 0,1,\ldots,20, \\ r_{i_1}(l') &= 0.4 + l'\cdot 0.01, \quad l' = 0,1,\ldots,40, \\ r_{i_2}(l'') &= 0.4 + l''\cdot 0.01, \quad l'' = 0,1,\ldots,40. \end{aligned}$$

Therefore, $\bar{h}_1$ and $\bar{h}_2$ are computed as:

$$\bar{h}_1(k_1) = \frac{1}{21}\sum_{l=1}^{21} h_1(k_1, r_1(l)), \quad (20)$$

$$\bar{h}_2(k_1,k_2) = \frac{1}{41^2}\sum_{l'=1}^{41}\sum_{l''=1}^{41} h_2(k_1,k_2,r_{i_1}(l'),r_{i_2}(l'')). \quad (21)$$

Since the kernels in (17) and (18) are governed by underdamped dynamics, it is more appropriate to adopt the two-parameter Kautz basis. In this example, the number of Kautz functions is chosen as $M = 8$. The central values of the expansion coefficients and their maximum deviations are computed using (10) and (11), respectively.

The maximum deviation $\Delta h_1(k)$ relative to the mean first-order Volterra kernel in (17) is computed as the absolute value of the maximum uncertainty from the mean kernel $\bar{h}_1(k)$ at each time instant. The same idea holds true for the second-order kernel in (18). Then

$$\Delta h_1(k_1) = \left|\max\{h_1(k_1,r_1) - \bar{h}_1(k_1)\}\right|,$$

$$\Delta h_2(k_1,k_2) = \left|\max\{h_2(k_1,k_2,r_{i_1},r_{i_2}) - \bar{h}_2(k_1,k_2)\}\right|.$$

Figures 1 and 2 illustrate the deviations $\Delta h_1(k_1)$ and $\Delta h_2(k_1,k_2)$, respectively.

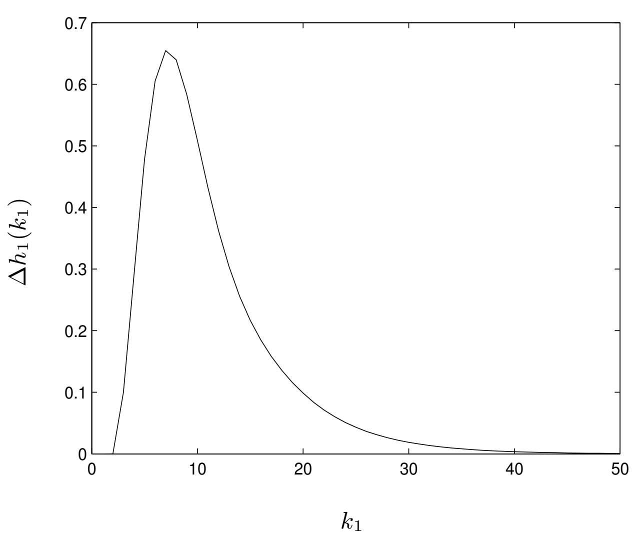

Fig. 1. Maximum deviation over the mean first-order kernel $\bar{h}_1(k_1)$.

Optimal values $\sigma^*_{i_1}$ and $\sigma^*_{i_1,i_2}$ are computed as proposed in Section IV-B, by solving the optimization problems (16)

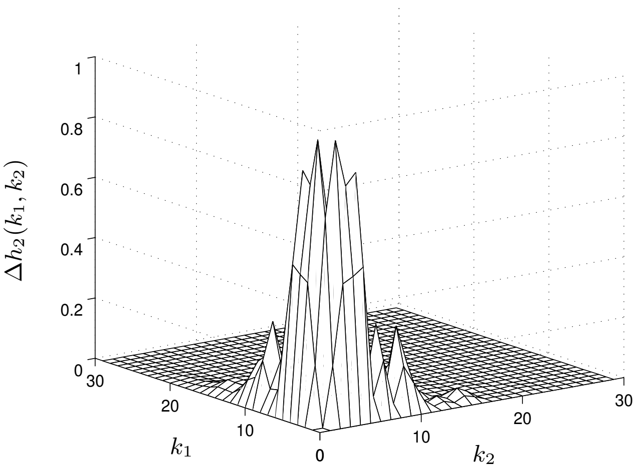

Fig. 2. Maximum deviation over the mean second-order kernel $\bar{h}_2(k_1, k_2)$.

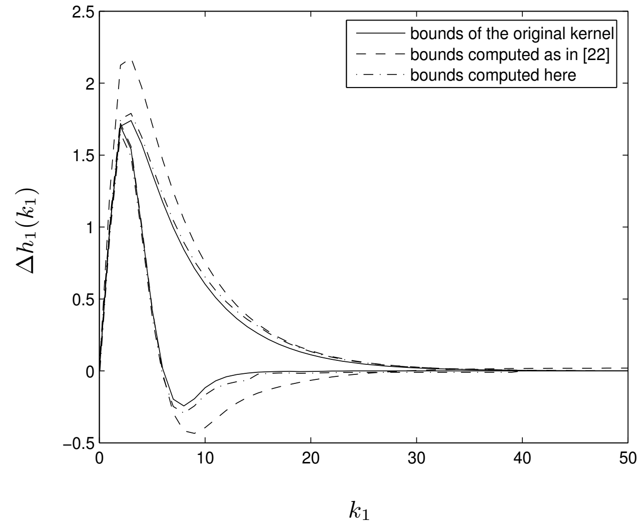

Fig. 3. Lower and upper bounds of kernel (17) using different approaches.

for $\eta = 1, 2$. The optimal solutions are:

$$\sigma^*_{i_1} = \begin{bmatrix} 0.7714 \\ 0.3474 \\ 0.5102 \\ 0.1147 \\ 0.0404 \\ 0.0288 \\ 0.3865 \\ 0.5512 \end{bmatrix}, \tag{22}$$

$$\sigma^*_{i_1,i_2} = \begin{bmatrix} 0.8180 & 0.8385 & 0.7948 & 0.8757 & 0.2844 & 0.4329 & 0.7833 & 0.4154 \\ 0.6602 & 0.5681 & 0.9568 & 0.7373 & 0.4692 & 0.2259 & 0.6808 & 0.3050 \\ 0.3420 & 0.3704 & 0.5226 & 0.1365 & 0.0648 & 0.5798 & 0.4611 & 0.8744 \\ 0.2897 & 0.7027 & 0.8801 & 0.0118 & 0.9883 & 0.7604 & 0.5678 & 0.0150 \\ 0.3412 & 0.5466 & 0.1730 & 0.8939 & 0.5828 & 0.5298 & 0.7942 & 0.7680 \\ 0.5341 & 0.4449 & 0.9797 & 0.1991 & 0.4235 & 0.6405 & 0.0592 & 0.9708 \\ 0.7271 & 0.6946 & 0.2714 & 0.2987 & 0.5155 & 0.2091 & 0.6029 & 0.9901 \\ 0.3093 & 0.6213 & 0.2523 & 0.6614 & 0.3340 & 0.3798 & 0.0503 & 0.7889 \end{bmatrix}. \tag{23}$$

Based on these numerical results, one can verify that the obtained kernel bounds comprise all the original (uncertain) kernels. Figure 3 illustrates the lower and upper bounds of the first-order kernel (17) computed using two different approaches, namely, the approach proposed in the present paper and the one that uses solely equations (14) and (15), as described in [22].

As previously mentioned in Section I, it can be seen in Figure 3 that the bounds computed as described in [22] have not been able to represent the bounds of h_1 for all time instants. In fact, beyond $k_1 = 29$ the lower bounds obtained by that method do not contain the lower bounds of the original kernel. The reason is that the mathematical condition $\psi_{1,i_1}(k_1) > 0$ cannot be satisfied for Kautz functions. Differently, the formulation proposed in the present paper circumvents this drawback by ensuring that the optimal bounds do contain both the lower and the upper bounds of the original kernels. This is a constraint of the optimization problems in (16).

Figure 4 compares the bounds obtained for the second-order kernel in (18) projected on the plane $k_1 = 10$. The method in [22] – extended to nonlinear Volterra models in [5] – has not been able to represent the bounds of h_2 for all time instants, in contrast to the method proposed here. Again, the reason is the same as that discussed above with respect to the 1st order kernel.

It has been verified numerically that, as expected, the models obtained from $\sigma^*_{i_1}$ in (22) and $\sigma^*_{i_1,i_2}$ in (23) are indeed able to represent the kernels (17) and (18), respectively. Mathematically, one has $\{h_1(k_1)\} \subseteq \{\breve{h}_1(k_1)\}$ and $\{h_2(k_1,k_2)\} \subseteq \{\breve{h}_2(k_1,k_2)\}$, thus satisfying the specifications imposed by the constraints of problem (16).

VI. Conclusions

The modeling of uncertain dynamic systems using an orthonormal basis function framework has been investigated. This problem has been addressed in the context of nonlinear Volterra models in which the bounds of the uncertain OBF model coefficients are estimated from a set of Volterra kernels with known structured uncertainties. This is an extension — to the context of nonlinear Volterra models — of previous

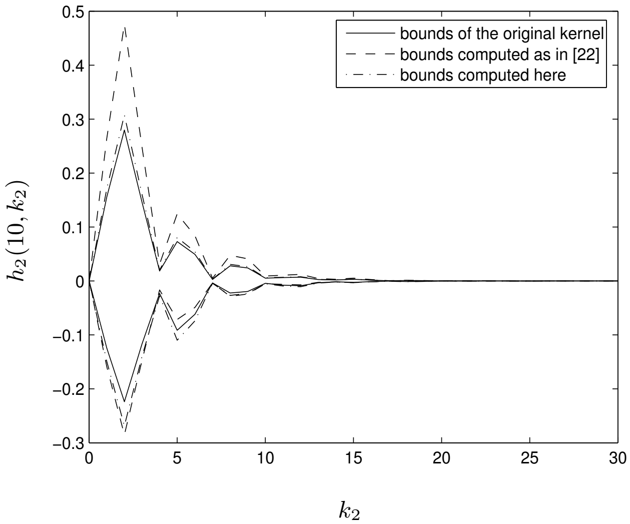

Fig. 4. Lower and upper bounds of kernel (18) using different approaches.

results for linear systems previously decribed in the literature

The proposed method may be useful in nonlinear robust predictive control, where the input signal is calculated considering the worst possible performance with respect to the tracking of the reference signal. A simulated example has been presented to illustrate the theoretical results. In this example, the modeling of first- and second-order uncertain Volterra kernels has been illustrated using the two-parameter Kautz functions, but other related orthonormal bases (e.g. Laguerre or GOBF) could have been used as well.

In future work, the authors intend to extend the proposed method towards models with uncertainties taking place at the output signal.

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