Improved MRI reconstruction and denoising using SVD-based low-rank approximation

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***Abstract*—The reconstruction of multi-dimensional magnetic resonance imaging (MRI) data can be a computationally demanding task. Signal-to-noise ratio is also a concern, specially in high-resolution imaging. Data compression may be useful not only for reducing reconstruction complexity and memory requirements, but also for reducing noise, as it is capable of eliminating spurious components. This work proposes the use of SVD-based low-rank approximation for the reconstruction and denoising of MRI data. The Akaike information criterion is used to estimate the appropriate model order. The model order is used to remove noisy components and to reduce the amount of data to be stored and processed. The proposed method is evaluated using *in vivo* MRI data. We present images reconstructed using less than 20% of the original data size and with a similar quality in terms of visual inspection. A quantitative evaluation is also presented.**

I. Introduction

Magnetic resonance imaging (MRI) of living human tissue started in the 1970s. Due to its recentness, MRI is a very fruitful area of research in the bioengineering and signal processing fields [1]. The general MRI process is illustrated in Figure 1. The image acquisition process can use one or multiple coils, different pulse sequences and gradient fields, and is currently a major field of research and work. The acquired data corresponds to the Fourier transform of the image, $A(k_x, k_y)$, also called k-space. The image, $A(x, y)$, is obtained using reconstruction algorithms, which are typically based on the inverse Fourier transform.

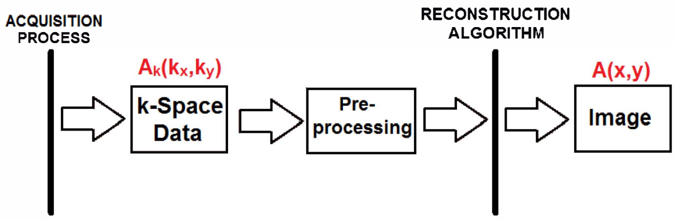

Figure 1. Magnetic resonance imaging acquisition and reconstruction process.

The data acquisition process usually focus on reducing scan time in order to increase patient comfort and throughput and to reduce motion artifacts. However, reduced scan time results in reduced signal-to-noise ratio (SNR). This issue is aggravated when high-resolution imaging is used, as a reduced voxel size is generally associated with significantly lower SNR. A post-acquisition approach to improve SNR is to use data statistics in order to identify and remove noise components.

Singular value decomposition (SVD) is a well-known technique for data compression [2] as well as for denoising [4]. Some applications have already been demonstrated for MRI, such as data reconstruction [3] and denoising [4]. In multi-channel MRI systems, SVD can also be used for coil compression [5]. SVD-based low-rank approximation can be applied either before image reconstruction — treating data in the k-space — or after reconstruction — dealing with the reconstructed image and aiming to use less memory to store it.

This paper proposes the use of model order selection and decomposition schemes for MRI data reconstruction, based on the Akaike information criterion (AIC) and singular value decomposition. By applying a model order selection, we calculate the number of components necessary to represent the MRI data. Then, SVD-based low-rank approximation is used to reconstruct the MRI image from the reduced number of components. In order to validate this technique, both image and k-space domain denoising are evaluated and compared by means of both signal-to-error ratio (SER) and visual inspection.

The remainder of this paper is organized as follows: Section II presents the mathematical formulation for MRI image reconstruction and for singular value decomposition. Section III introduces the problem and explains the experimental methodology. Section IV describes the proposed approach, and explainins how SVD and AIC are used to achieve the desired results. Section V presents the results of truncated SVD applied to an MRI brain image. Section VI presents a final analysis of the results and this work's conclusions.

II. Mathematical Formulation

In this section, we present the theoretical fundamentals of MRI image reconstruction and the principles of singular value decomposition.

A. Magnetic resonance imaging

The acquired MRI signal at a particular time instant t corresponds to a sample of the Fourier transform $A(k_x, k_y)$

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of the cross-sectional image $A(x, y)$:

$$A(k_x, k_y) = \int_x \int_y A(x, y)\, e^{-j2\pi(k_x x + k_y y)}\, dx\, dy. \quad (1)$$

The Fourier coordinates k_x and k_y vary with time, according to:

$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau)\, d\tau \quad (2)$$

$$k_y(t) = \frac{\gamma}{2\pi} \int_0^t G_y(\tau)\, d\tau, \quad (3)$$

where γ is the gyromagnetic ratio (for hydrogen protons, γ = 42.57 MHz/T), and $G_x(y)$ and $G_y(y)$ are time-varying magnetic field gradients along the x- and y-axes, respectively.

The $A(x, y)$ image is reconstructed using a two-dimensional inverse Fourier transform along k_x and k_y. The k-space data is typically digitized and reconstructed on a computer. Therefore, the reconstructed MRI image corresponds to a matrix of grayscale pixel-intensity values, A.

B. Singular value decomposition

Considering the image matrix A, which is an $M \times N$ matrix of data, it is possible to obtain its singular values and singular vectors according to:

$$A = USV^{\mathrm{T}} \quad (4)$$

$$= \sum_{i=1}^{N} \sigma_i u_i v_i^{\mathrm{H}}, \quad (5)$$

where U is an $M \times M$ matrix, S is an $M \times N$ diagonal matrix, and V^{T} is an $N \times N$ matrix. After calculating each singular value and singular vector of the matrix, we are able to decompose it using singular value decomposition.

Considering that the rank of the signal matrix A is r, and that $M \geq N$, then $r \leq N$. Equation (4) describes the singular value decomposition of A [6], [7].

The columns of U are called the left singular vectors, $\{u_K\}$, and the rows of V^T correspond to the right singular vectors, $\{v_K\}$. The elements of the diagonal matrix S are called the singular values. The singular values contain information about the importance of each vector — left singular and right singular — in spanning the signal space (or generating the MRI data, in the proposed application). The singular values describe how essential each component is for data description.

Furthermore, $s_K > 0$ for $1 \leq K \leq r$, and $s_i = 0$ for $(r + 1) \leq i \leq N$, and the singular values are ordered from the highest to the lowest. Therefore, this composition indicates that the singular values with the lowest indexes represent the more important components of the signal. Thus, it is possible to apply an algorithm in order to determine the model order and therefore reduce the number of components in the matrices U, S, and V in order to recover the original data without significant loss of information, and possibly reduce noise [7].

In this paper, we propose the use of this decomposition scheme for reducing MRI data.

In order to correctly validate our analysis, we compare the results of SVD decomposition schemes considering the original signal (in k-space) and the reconstructed signal (in image domain).

III. Problem statement

By applying the AIC, the amount of signal singular values and its respective left and right singular vectors is estimated. Therefore, it is possible to use this number, also called model order, to represent the original data.

We call principal components (PC) the singular values and singular vectors related to the model order. Using the PC, one can increase the signal-to-error ratio in the reconstruction by removing noisy components and, therefore, enhance the reconstructed image quality. Another interesting point is that with fewer components, the data require less physical memory for storage and/or less bandwidth for transmission.

IV. Proposed SVD-based low-rank approximation via Akaike information criterion

The use of decomposition schemes enables us to select data from a given set of information and therefore identify which components are the most important for describing the original element. Data selection is a way of reducing noise from the acquired data based on its statistics and the distribution of its components. In this work, we analyze the singular value profile, obtained by singular value decomposition of any kind of complex rectangular matrices.

From the singular value matrix, it is possible to identify the model order and therefore apply a selection algorithm that deals only with the important information, i.e., the information that is required for characterizing the original data and necessary for accurate image reconstruction. Furthermore, one can identify which singular value components correspond to noise, and remove them before recomposing the original image.

The optimal selection of the number of components is performed using a selection criterion, or — in image analysis — by visual inspection. However, the latter is a less general solution, because it is sensitive to subjective analysis and differs from person to person. Therefore, this option should be used only to refine the final result. In this work, the number of principal components is selected using an objective approach: the Akaike information criterion (AIC).

A. Akaike information criterion

The Akaike information criterion is used to select the necessary number of components to describe a signal without loss of information. It is a mathematical criterion based on information theory, in which given a set of candidate models for a data, the preferred model is the one with the minimum AIC value, where the AIC value is given by [9], [10]:

$$\mathrm{AIC} = 2k - 2\ln(L), \quad (6)$$

where k is the number of parameters in the statistical model, and L is the maximized value of the likelihood function for the estimated model.

For the proposed analysis, the AIC number is calculated according to [8]:

$$\text{AIC(m)} = -N\cdot(M-m)\cdot\log\left(\frac{g(m)}{a(m)}\right)+m\cdot(2M-m), \quad (7)$$

where M and N correspond to the image size, and m is the number of considered components — and therefore the value that must be found in order to minimize the expression —, $g(m)$ is the geometric mean of the m smallest eigenvalues of the data and $a(m)$ is the arithmetic mean of m smallest singular values. Note that the singular values are the eigenvalues are the square of the singular values.

After finding the value of m that minimizes equation (7), it is possible to find the model order, which is given by: $\hat{D} = M - m$. The model order provides information about the minimum number of components that accurately represents the signal, and, therefore, allows us to discard the other components, which are considered noisy elements. Thus, this algorithm is used to reduce the amount of data used for image reconstruction, and also for denoising.

In this section, the combination of AIC model order selection and low-rank truncated singular value decomposition is applied to the data in both k-space and image domains. The results are compared both qualitatively (by means of visual inspection) and quantitatively (by calculating the signal-to-error ratio).

B. Using the estimated model order for SVD-based low-rank approximation

After finding the optimal model order, we can reduce the singular value and singular vector matrices by selecting only the components that are important for accurately describing the signal. For example, given the matrix A of size $M \times N$, which is originally described as stated in equation (4) — where $U_{M\times M}$, $S_{M\times N}$, and $V_{N\times N}$ —, then if the model order is D ($1 \leq D \leq \min(M,N)$), we can use only the highest D singular values and the first D left and right singular vectors to represent the original data. Therefore, we have:

$$\begin{aligned} A &= U_s S_s V_s^T &(8)\\ &= \sum_{i=1}^{D} \sigma_i u_i v_i^H, &(9) \end{aligned}$$

where U_s and V_s contain the first D columns of U and V, respectively, and S_s is the diagonal matrix with the D highest singular values. The process is also called low-rank approximation or low-rank truncation.

In the next section, we evaluate the use of truncated SVD for representing single-coil, two-dimensional MRI data. The information is therefore matricial. SVD was applied in both k-space and image domains, so that it is possible to identify which k-space or image components are more likely to represent noisy elements. Using the AIC, the singular value matrices are reduced to represent only meaningful and useful data. Finally, image reconstruction is performed and the results are compared both qualitatively and quantitatively.

V. Results

The data used in this work was downloaded at http://shorty.usc.edu/class/591/fall04/. Figure 2 shows the image reconstructed from these data, using all components of its singular value matrix. Reconstruction was performed using an inverse two-dimensional discrete Fourier transform. This result will be used as ground-truth reference.

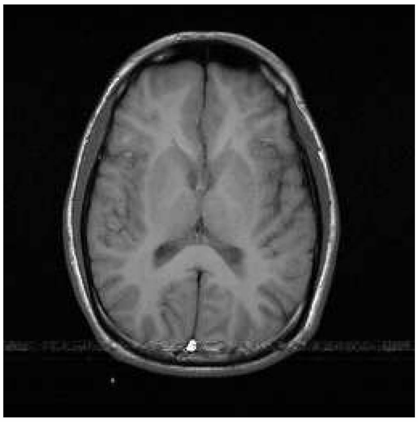

Figure 2. Ground-truth reference image obtained using all components of its singular value matrix.

The image show in Figure 2 is a 256×256 complex matrix, which generates 256×256 matrices U, S, and V. In order to verify the efficiency of the proposed method, random noise is added to the original data (in k-space) — which is already originally noisy due to the acquisition process and channel inhomogeneities —, and then SVD is applied. Noise addition is done in both real and imaginary values — corresponding to the image — according to:

$$X_k(k_x,k_y) = A_k(k_x,k_y) + n(k_x,k_y) \quad (10)$$

$n(k_x,k_y)$ are the zero mean i.i.d. additive Gaussian noise samples with variance $\sigma_s^2 = 9$.

The number of singular values used for reconstruction is selected both by using the Akaike information criterion and by visual inspection, in order to perform quantitative and qualitative analyses.

After applying the SVD, we obtain the singular value profiles for both k-space and image domains (Figures 3 and 4, respectively). Both profiles are very similar, thus the two approaches present similar results. According to these profiles, the most important singular values lie in the first 50 components of data. Thus, by this simple analysis, it is possible to state that the singular values matrix which is

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originally a 256×256 matrix, can be reduced to a 50×50 matrix, without significant loss of information. This kind of reduction means dealing with less than one fifth of the data in each dimension of matrix S. Then, both U and V matrices can also be reduced and less data would be stored.

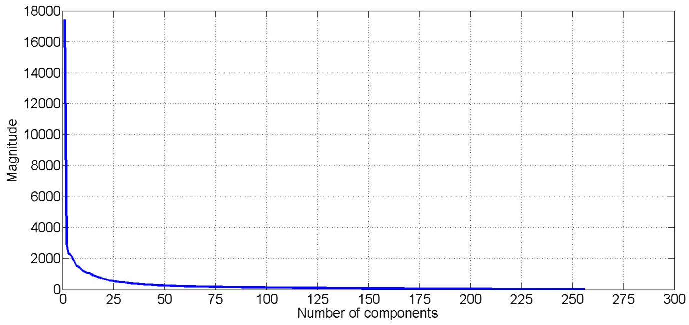

Figure 3. Singular value profile in k-space (Fourier) domain.

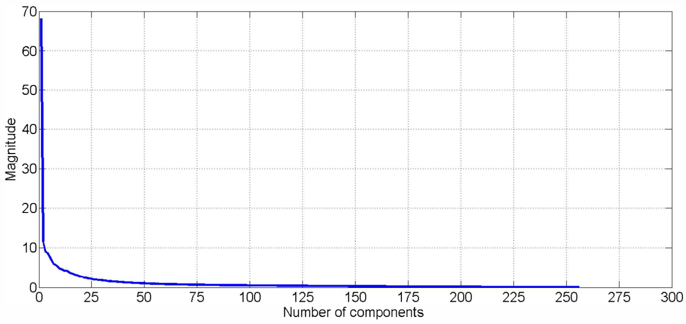

Figure 4. Singular value profile in image domain.

The reconstruction root mean squared error (RMSE) as a function of the number of components for k-space domain SVD is shown in Figure 5. With 50 components, the RMSE is about -11 dB. This is a good threshold to be used in the proposed reconstructions.

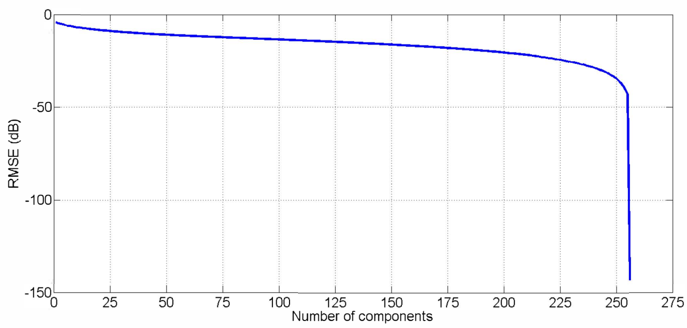

Figure 5. Root mean squared error as a function of the number of components, for k-space domain analysis.

When applying the AIC for k-space domain, the selected number of components to be used for data reconstruction is 30. Therefore, matrices U, S, and V are 256×30, 30×30, and 256×30, respectively. The reconstructed image obtained using 30-coefficient truncated SVD in k-space domain is shown in Figure 6. The error between this result and the ground-truth reference is shown in Figure 7. Even though the error image presents coherent brain patterns, the SER is approximately 18.8 dB — which is calculated as follows:

$$\mathrm{SER(dB)} = 10 \cdot \log_{10} \frac{\sum_{k_x} \sum_{k_y} (|X_k(k_x, k_y)|^2)}{\sum_{k_x} \sum_{k_y} (|X_k(k_x, k_y) - A_k(k_x, k_y)|^2)} \tag{11}$$

where $X_k(k_x, k_y)$ is the ground-truth image, $A_k(k_x, k_y)$ is the reconstructed one and the operator $|\cdot|$ returns the magnitude.

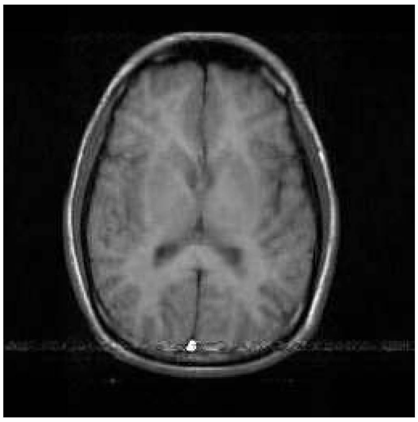

Figure 6. Reconstructed image obtained using the proposed AIC/SVD approach in k-space domain.

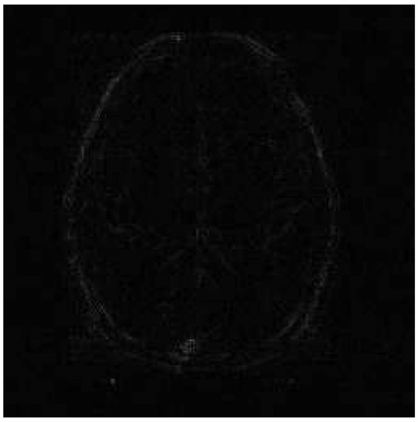

Figure 7. Error between the result shown in Figure 6 and the ground-truth reference, shown in Figure 2.

Applying the same process in image domain, it is possible to determine the number of essential image components. The AIC-determined number of components was the same: 30. The image obtained using 30-coefficient truncated SVD in image domain is shown in Figure 8. As expected, both results are very similar. The maximum difference between k-space and image-domain reconstructed images is on the order of 10^{-14}, which may be considered quantization error. Therefore, the two results can be considered equal.

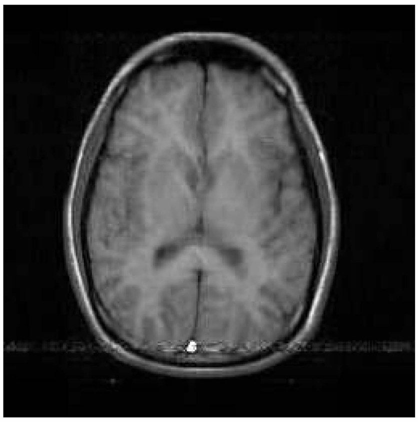

Figure 8. Reconstructed image obtained using the proposed AIC/SVD approach in image domain.

Figure 9 shows the results using only 10 components, for both k-space and image-domain approaches. Both reconstructions present bad results, and the denoising process is not effective. Here, meaningful data is interpreted as noise, and the error image is completely coherent with the actual image, which suggests that important components were discarded.

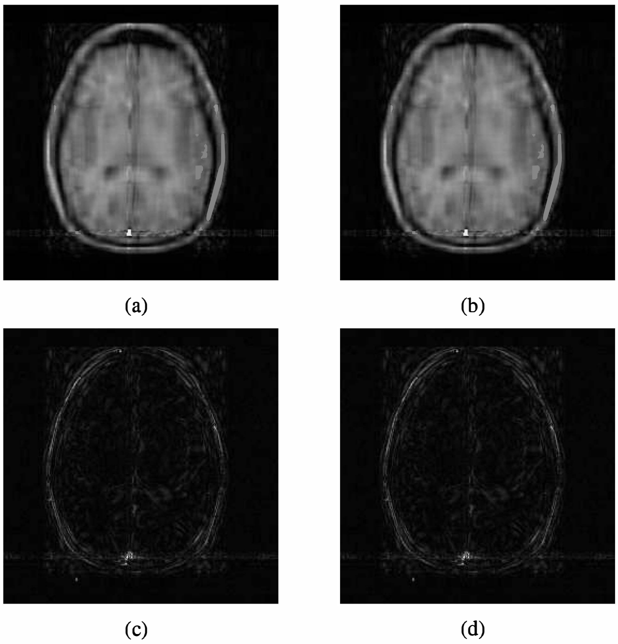

Figure 9. Images obtained using only 10 SVD components: (a) reconstruction in k-space; (b) reconstruction in image domain; (c) error between reference image and k-space result; and (d) error between reference image and image-domain result.

By comparing k-space domain and image domain truncated SVD results in terms of signal-to-error ratio, it can be concluded that both approaches are equivalent (Table 1). Thus, it is possible to implement the algorithm using either the k-space or image domain approaches, without loss of reconstruction quality. The results in Table 1 also show that the proposed approach performs quantitatively well, as the SER is higher than 10 dB even with a small number of components.

Table I
SIGNAL-TO-ERROR RATIO (SER) FOR IMAGE AND FREQUENCY DOMAINS AND ROOT-MEAN SQUARED ERROR (RMSE) FOR VARYING NUMBER OF COMPONENTS (FROM 10 TO 130).

Number of Components	SER (dB) frequency domain	SER (dB) image domain	RMSE (dB)
10	13.4	13.4	−6.7
22	17.1	17.1	−8.6
31	19.0	19.0	−9.5
43	20.9	20.9	−10.5
55	22.4	22.4	−11.2
67	23.7	23.7	−11.9
79	24.9	24.9	−12.5
88	25.8	25.8	−12.9
97	26.7	26.7	−13.3
109	27.9	27.9	−13.9
118	28.8	28.8	−14.4
124	29.4	29.4	−14.7
130	30.1	30.1	−15.0

When using the AIC to select the number of components — 30, for both image and k-space domains — the SER is 18.8 dB, which is an acceptable result. However, when computational effort is not an issue, it is possible to use a higher number of components in order to achieve higher SER. For example, using 128 components (half of the original number) will result in greater computational complexity, because larger matrices are used; however, the SER will be 29.8 dB.

Using a qualitative analysis, it is possible to select points where data becomes visually similar to the original image and where it is achieved a visually accurate image. In this analysis, these points correspond to 10 and 28 components respectivelly.

Lastly, the memory storage as a function of the number of components is shown in Figure 10. Using the AIC, the reconstruction process uses 11.7% of the memory used to reconstruct the image with all components. However, the result is very similar to the ground-truth reference, as previously shown. Then, in addition to reducing noise in the images, the use of SVD reduces the amount of memory necessary to store the data. Therefore, it may be useful when dealing with high-resolution, multi-channel and/or multi-dimensional MRI data.

VI. CONCLUSION

The results suggest that truncated SVD works for MRI data and can improve its reconstruction by reducing noise and making the reconstruction process less computationally demanding. Using SVD-based low rank approximation, the process requires a small number of components, and, therefore, less memory and a reduced number of computational operations. The results also showed that reconstruction after denoising in k-space and image domains are equal for both quantitative and qualitative analysis.

The use of a larger number of components generally presents better results. However, when using all components the image is reconstructed with its noisy elements. Therefore, it is interesting to perform model order selection and singular

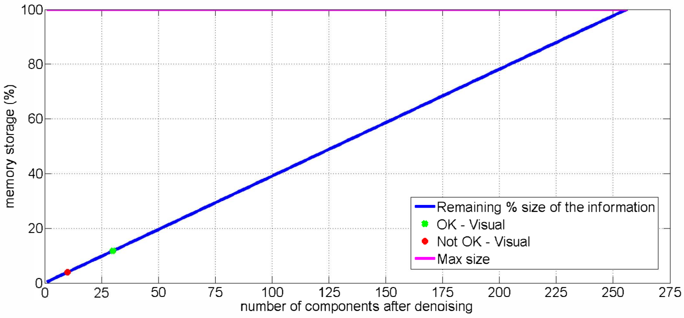

Figure 10. Memory storage necessary for data reconstruction, according to the number of components selected using the AIC. The red circle indicates where image initially can be seen but it is not well represented — using only 10 components — and the green × corresponds to a more accurate representation, obtained with the minimum number of components according to the AIC, i.e., 30.

value decomposition low-rank truncation in order to represent the data with only the principal components.

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