

ENE0167 - Controle Digital

Notas de Aula, ref. Franklin, 1998

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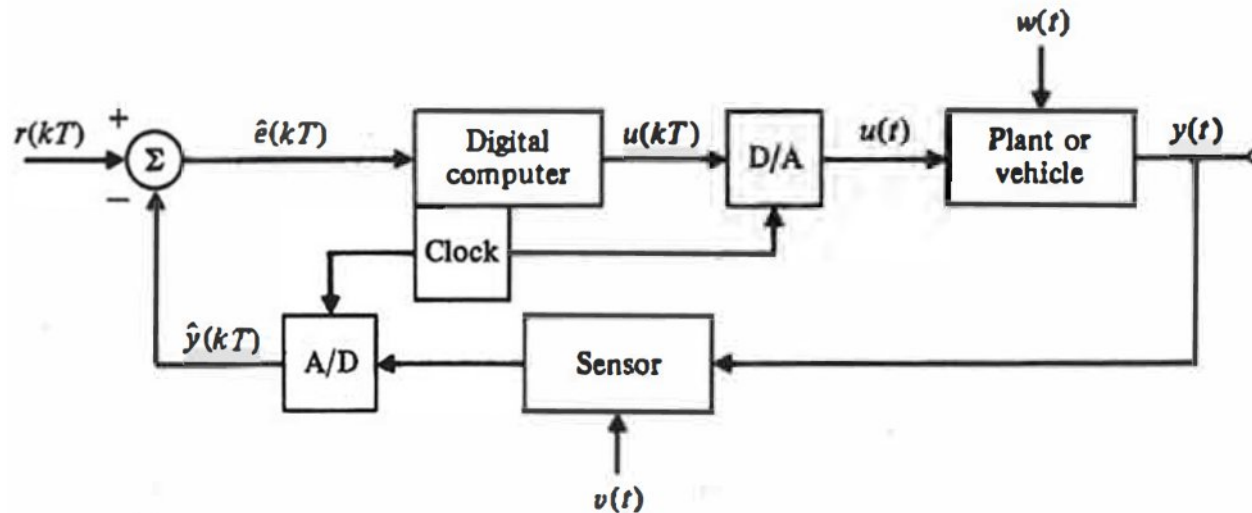
A leitura das seções em livro texto, dos conteúdos vistos nestes slides, é essencial ao entendimento da disciplina.

As Listas de Exercícios L1 e L2 (Franklin) ajudam a fixar o conteúdo.

Provas anteriores ilustram a forma com a qual o conteúdo é cobrado.

Os slides são o suporte para o desenvolvimento em sala - importante tomar nota, em sala, do que é "interessante"!

1. Introdução — Sistemas de Controle Digital



Notation:

- r = reference or command inputs
- u = control or actuator input signal
- y = controlled or output signal
- $\hat{y}$ = instrument or sensor output, usually an approximation to or estimate of y . (For any variable, say θ , the notation $\hat{\theta}$ is now commonly taken from statistics to mean an estimate of θ .)
- e = $r - \hat{y}$ = indicated error
- c = $r - y$ = system error
- w = disturbance input to the plant
- v = disturbance or noise in the sensor
- A/D = analog-to-digital converter
- D/A = digital-to-analog converter

Conversor A/D

- Quantização

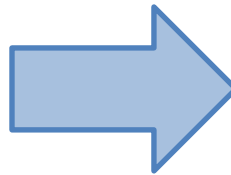
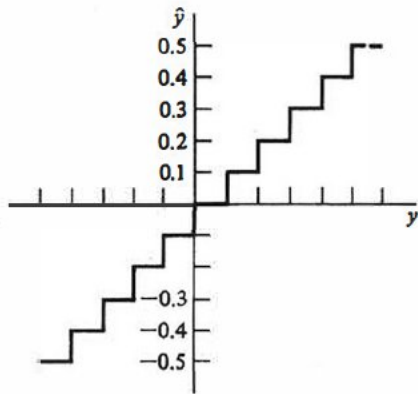
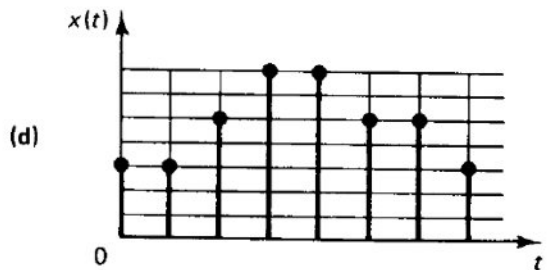
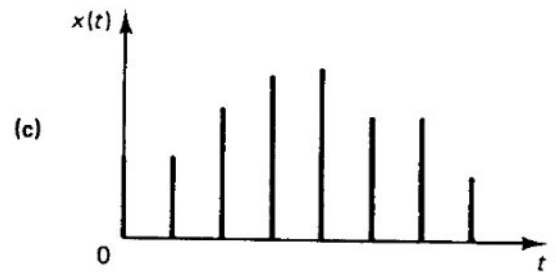
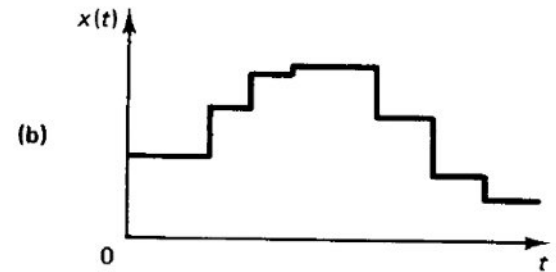
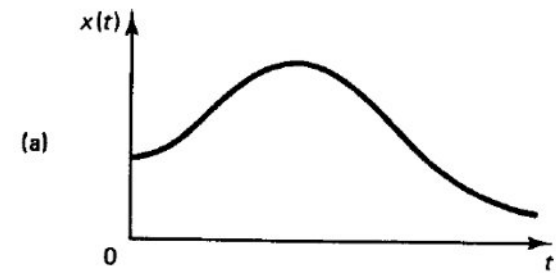
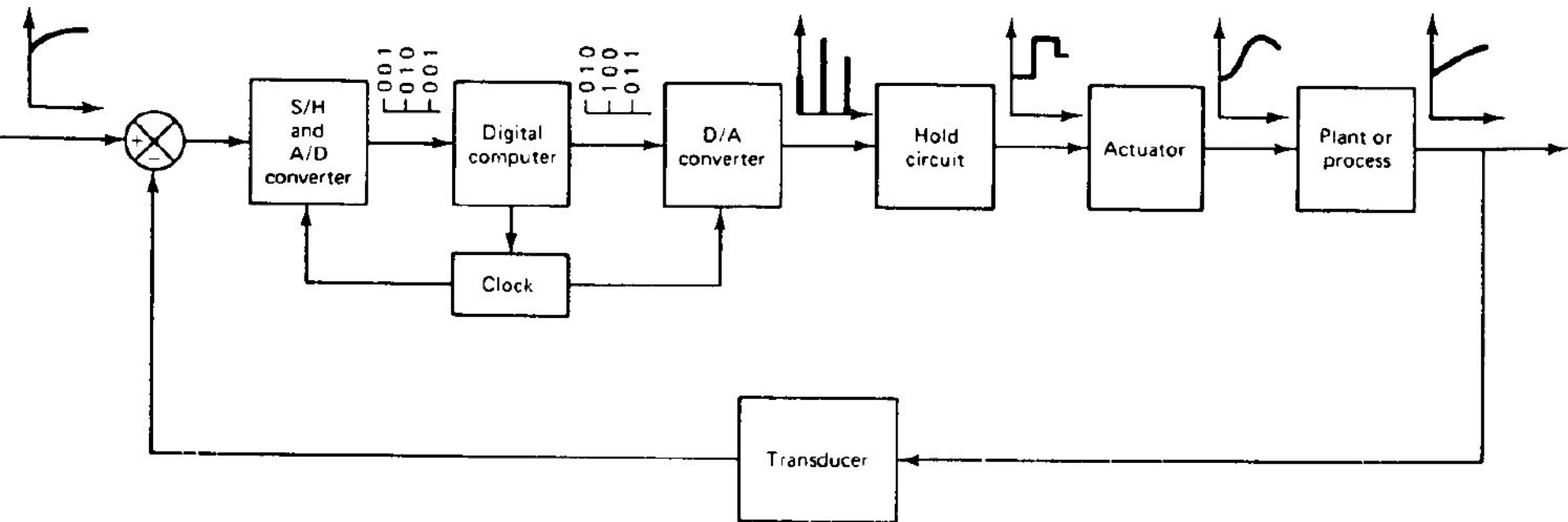


Figure 1-1 (a) Continuous-time analog signal; (b) continuous-time quantized signal; (c) sampled-data signal; (d) digital signal.



Sinais - Sistema de Controle Digital

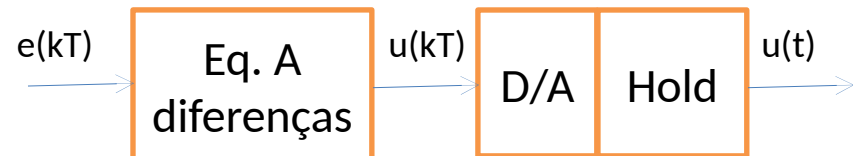
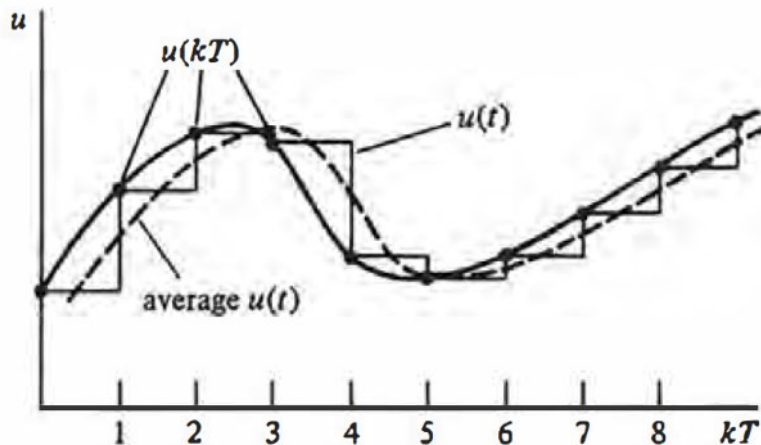


Obs: Reference and error detector can be implemented as software.
(Ogata, Discrete Control)

3.2 Amostragem

- Hold ➡

Additional delay in a feedback controlled system reduce stability and damping factor (overshoot $\uparrow$).



Atraso médio $T/2 \rightarrow$
análise do sistema contínuo com Hold

$$G_h(s) = \frac{2/T}{s + 2/T} \quad (\text{Padé})$$

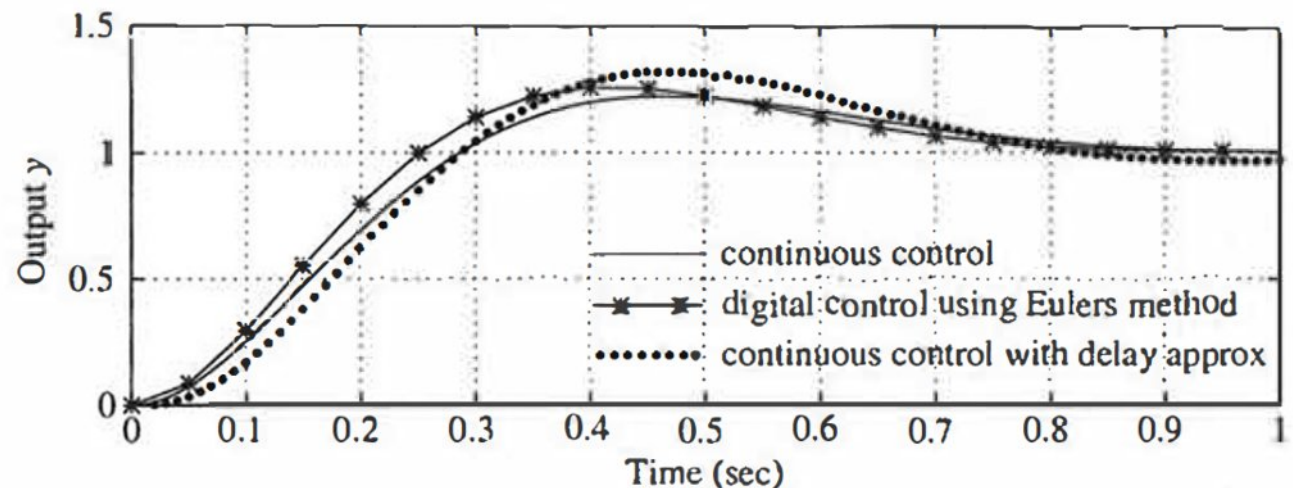
Análise no domínio- ω

- Hold: avg. delay in time, $T/2 \rightarrow \exp(-j\omega T/2)$

Phase decrease $\delta \phi = \frac{-\omega T}{2}$

Figure 3.4

Continuous and digital step response at 20 Hz sample rate showing results with a $T/2$ delay approximation



Exemplo 3.3

$$G(s) = \frac{1}{s(s+1)}$$

$$D(s) = 70 \frac{s+2}{s+10}$$

$$G_h(s) = \frac{2/T}{s+2/T}$$

$$\text{sys0}(s) = \text{feedback}(G * D, 1)$$

$$\text{sys1}(s) = \text{feedback}(G * D * G_h, 1)$$

$$\text{sys2}(s) = \text{feedback}(G * D * \exp(-j\omega T/2), 1)$$

→ ML

```
g=tf(1,[1 1 0]),d=zpk(-2,-10,70)
```

```
gh=tf(20,[1 20])
```

```
gdd=c2d(g*d,.1)
```

```
sysc=feedback(g*d,1)
```

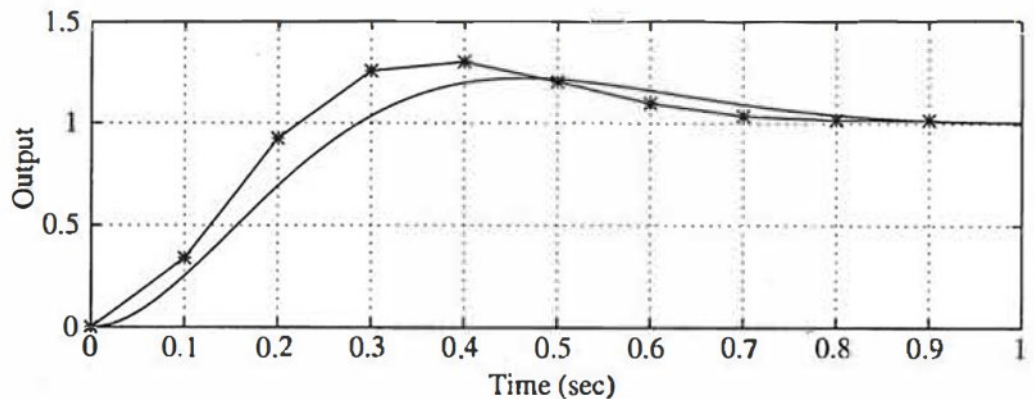
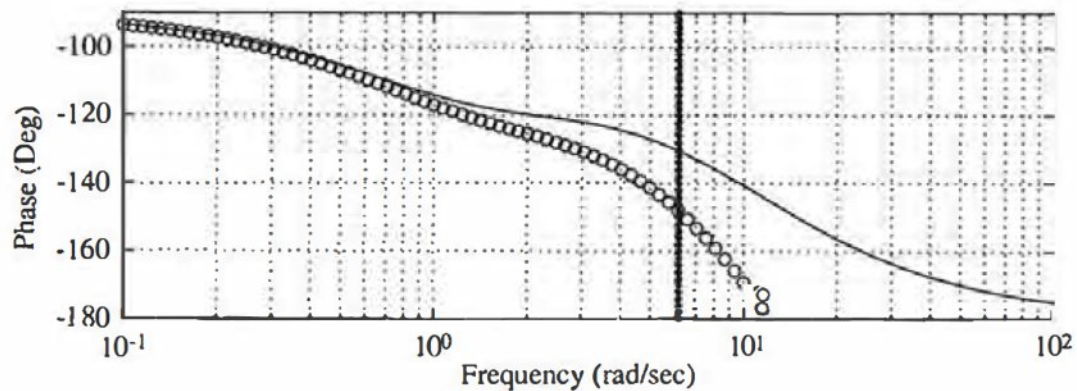
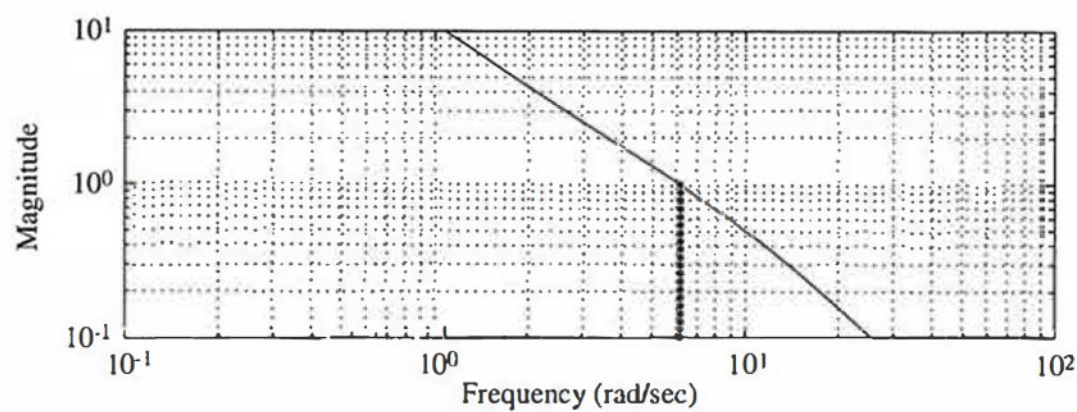
```
sysd=feedback(gdd,1)
```

```
sc=feedback(g*gh*d,1)
```

```
cc=d2c(sysd)
```

```
figure(2);
```

```
step(sysc,sysd,feedback(g*gh*d,1),cc)
```



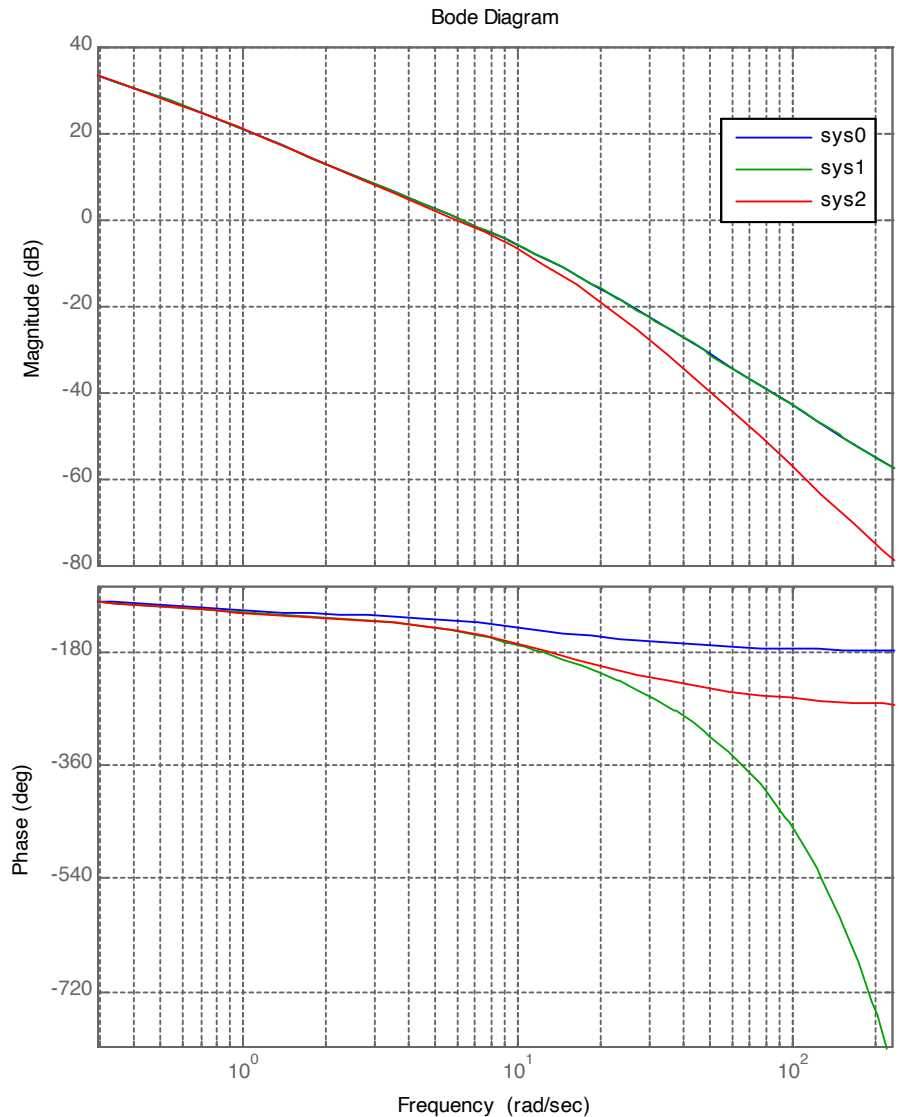
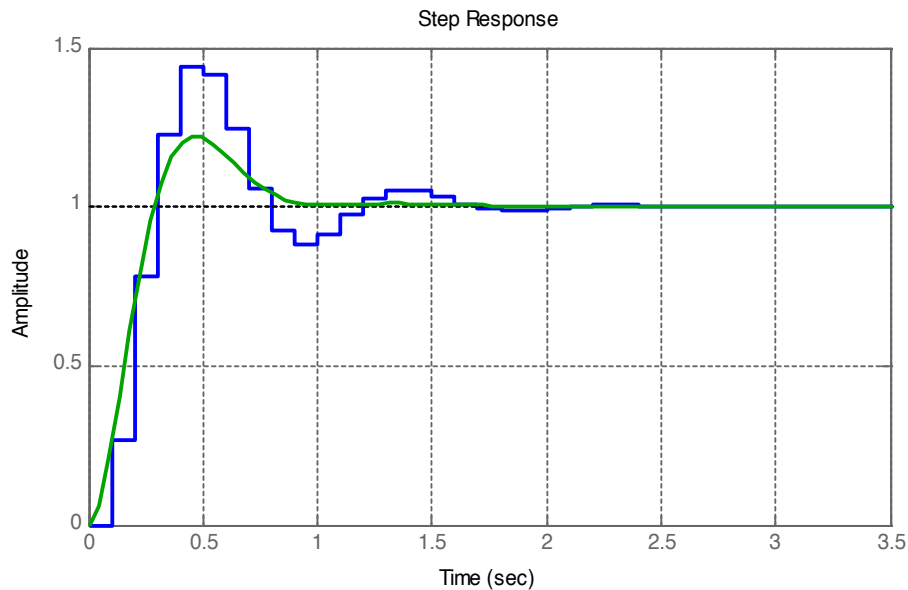
Exemplo 3.3

$$G(s) = \frac{1}{s(s+1)}; D(s) = 70 \frac{s+2}{s+10}; G_h(s) = \frac{2/T}{s+2/T}$$

$$\text{sys0}(s) = \text{feedback}(G * D, 1)$$

$$\text{sys1}(s) = \text{feedback}(G * D * G_h, 1)$$

$$\text{sys2}(s) = \text{feedback}(G * D * \exp(-j\omega T / 2), 1)$$



Controle PID por *emulação*

Digitalização pelo método de Euler $\dot{x} = \frac{x(k+1) - x(k)}{T}$

$$P : u(t) = Ke(t) \quad \longrightarrow \quad u(k) = Ke(k)$$

$$I : u(t) = \frac{K}{T_i} \int_0^t e(\eta) d\eta \quad \xrightarrow{\text{Euler}} \quad u(k) = u(k-1) + \frac{K}{T_i} e(k)$$

$$D : u(t) = \frac{K}{T_d} \dot{e}(t) \quad \xrightarrow{\text{Euler}} \quad u(k) = \frac{KT_d}{T} [e(k) - e(k-1)]$$

$$D(s) = \frac{u(s)}{e(s)} = K \left(1 + \frac{1}{T_i s} + T_d s \right) \Rightarrow \dot{u} = K \left(\dot{e} + \frac{e}{T_i} + T_d \ddot{e} \right)$$

PID discreto

Projeto Contínuo $\rightarrow$ discreto: $\sim 30\text{BW}$, quase-contínuo

$$u(k) = u(k-1) + K \left[\left(1 + \frac{T}{T_i} + \frac{T_d}{T} \right) e(k) - \left(1 + 2 \frac{T_d}{T} \right) e(k-1) + \frac{T_d}{T} e(k-2) \right]$$

4 – Análise de Sistemas Discretos

Computer as Dynamic Controller:

$$u_k = f(e_0, \dots, e_k; u_0, \dots, u_{k-1})$$

Linear Case, finite: recurrence equation

$$u_k = -a_1 u_{k-1} - a_2 u_{k-2} - \dots - a_n u_{k-n} + b_0 e_k + b_1 e_{k-1} + \dots + b_m e_{k-m}$$

Origem do nome:

$$\nabla u_k = u_k - u_{k-1}$$

$$\nabla^2 u_k = \nabla u_k - \nabla u_{k-1}$$

$$\nabla^n u_k = \nabla^{n-1} u_k - \nabla^{n-1} u_{k-1}$$



$$u_k = u_k$$

$$u_{k-1} = u_k - \nabla u_k$$

$$u_{k-2} = u_k - 2\nabla u_k + \nabla^2 u_k$$

Ex.: Difference . eq. 2.^a order
($b_0 = b_2 = 0$)

$$a_2 \nabla^2 u_k - (a_1 + 2a_2) \nabla u_k + (a_2 + a_1 + 1) u_k = b_0 e_k$$

4.2 A Função de Transferência Discreta

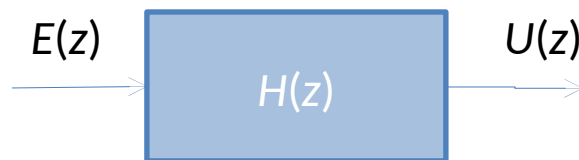
Transformada $\mathcal{Z}$ de $e_0, e_1, \dots, e_k, \dots$ $E(z) \triangleq \mathcal{Z}\{e(k)\}$
 $\triangleq \sum_{k=-\infty}^{\infty} e_k z^{-k}, \quad r_o < |z| < R_o$

Exemplo: $e^{-at} 1(t) \rightarrow e_k = e^{-akT} 1(kT)$

$$\begin{aligned} \sum_{k=-\infty}^{\infty} e_k z^{-k} &= \sum_0^{\infty} e^{-akT} z^{-k} \\ &= \sum_0^{\infty} (e^{-aT} z^{-1})^k \\ &= \frac{1}{1 - e^{-aT} z^{-1}} \\ &= \frac{z}{z - e^{-aT}} \quad |z| > e^{-aT} \end{aligned}$$

A Função de Transferência Discreta

$$H(z) \triangleq \frac{U(z)}{E(z)}$$



$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}}$$

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \dots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}$$

$$H(z) = \frac{b(z)}{a(z)}$$

$$U(z) = H(z) E(z)$$

Pólos e Zeros

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \dots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n} = \frac{b(z)}{a(z)}$$

$$H(z) = K \frac{\prod_{i=1}^m (z - z_i)}{\prod_{i=1}^n (z - p_i)}$$

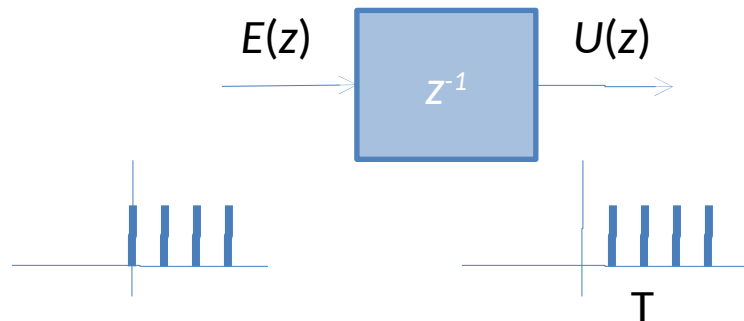
MatLab: `>> H = tf([b0 b1 ... bm],[1 a1 a2 ... an],T)`

`>> H2 = zp(z, p, k, T)`

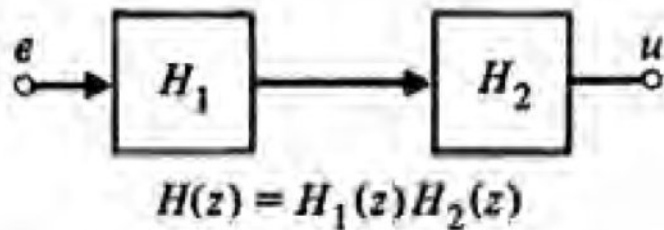
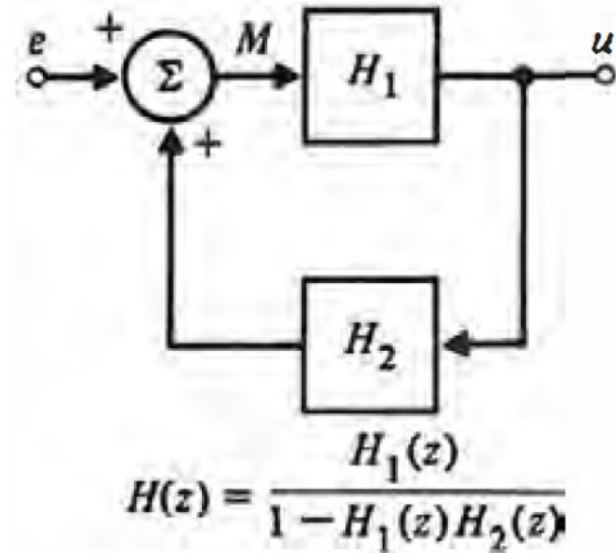
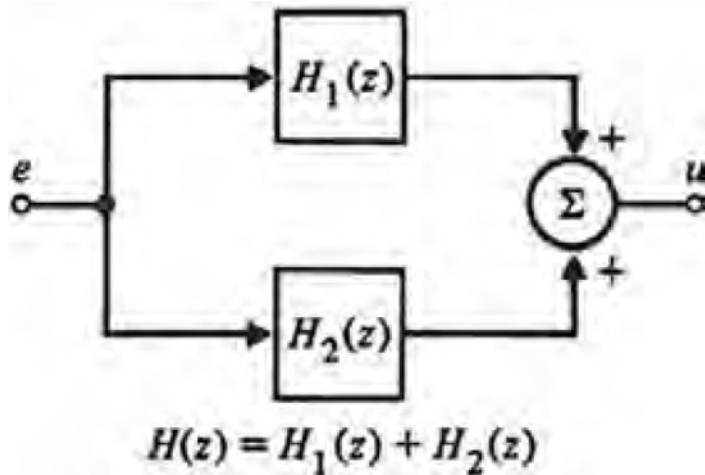
`>> H2 = zp(H)`

`>> H = tf(H2)`

Ex: degrau unitário $H(z) = z^{-1} \rightarrow u_k = e_{k-1}$

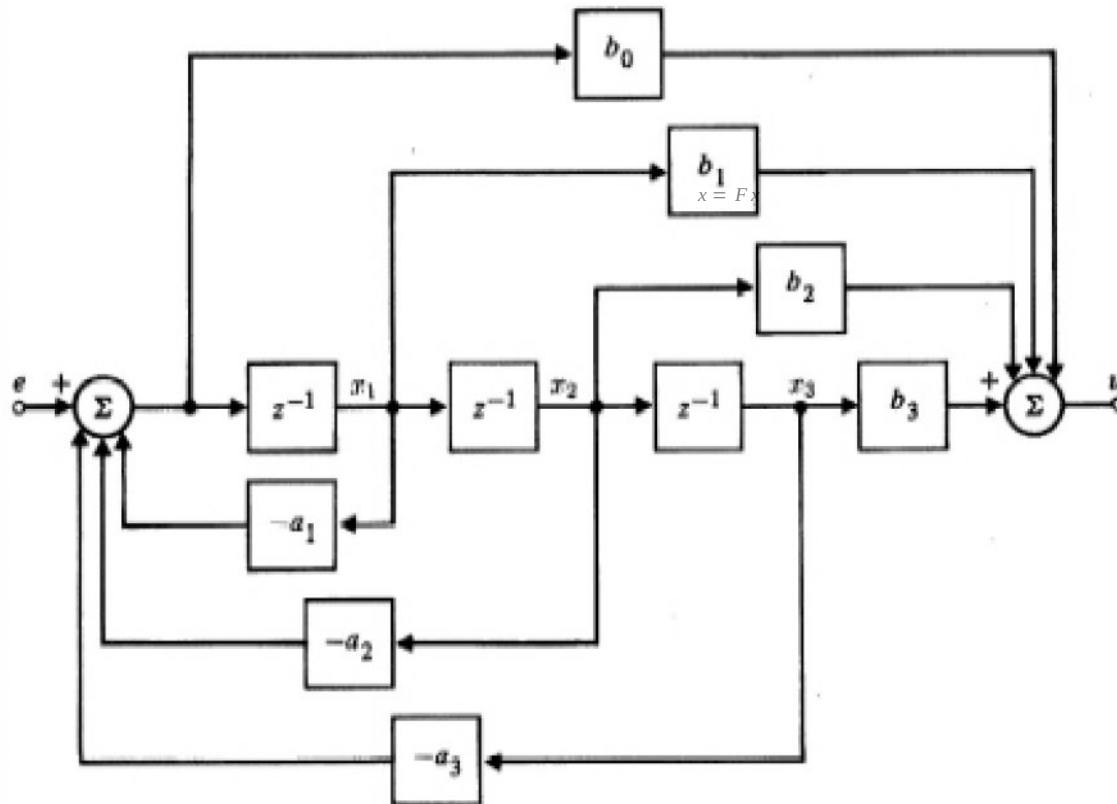


Diagramas de Blocos



4.3.3 Espaço de Estados

Forma Canônica Controlável



Franklin:

$$\dot{\mathbf{x}} = \mathbf{F} \mathbf{x} + \mathbf{G} u + \mathbf{G}_w w$$

$$y = \mathbf{H} \mathbf{x} + \mathbf{J} u$$

Ogata:

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} u + \mathbf{G}_w w$$

$$y = \mathbf{C} \mathbf{x} + \mathbf{D} u$$

$$\mathbf{x}(k+1) = \mathbf{A}_c \mathbf{x}(k) + \mathbf{B}_c e(k),$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{A}_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Forma Canônica Controlável

$$\frac{Y(k)}{U(k)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} + b_{-1}$$

b_{-1} é um "processo paralelo",
não influenciado pela dinâmica $[x_1 \ x_2 \ x_3]^T$

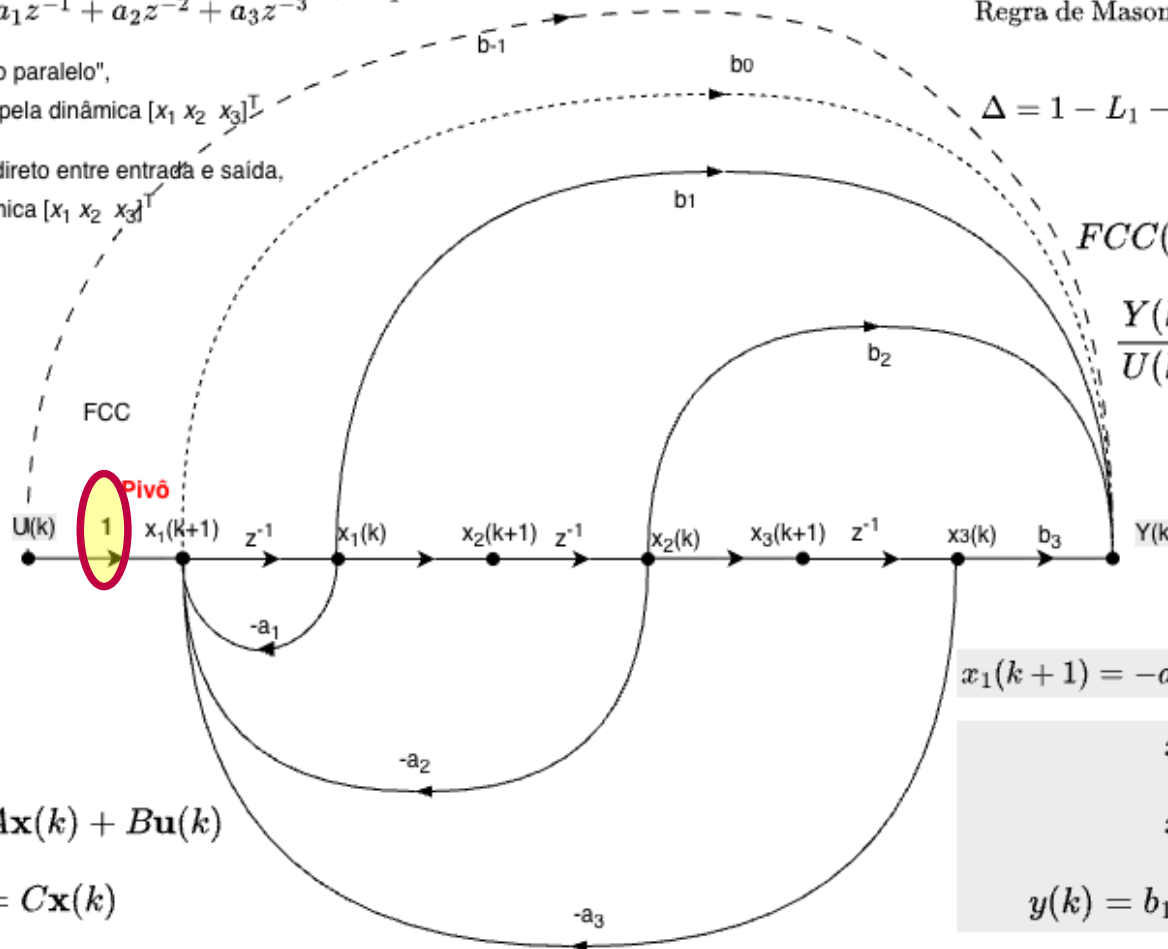
b_0 é um caminho direto entre entrada e saída,
afetado pela dinâmica $[x_1 \ x_2 \ x_3]^T$

Regra de Mason: $\frac{\sum_i^N P_i \Delta_i}{\Delta}$

$$\Delta = 1 - L_1 - L_2 - L_3 + L_1 L_2 + \dots - L_1 L_2 L_3 - \dots$$

FCC (assumindo $b_{-1} = b_0 = 0$)

$$\frac{Y(k)}{U(k)} = \frac{P_1 + P_2 + P_3}{1 - L_1 - L_2 - L_3}$$



$$x_1(k+1) = -a_1 x_1(k) - a_2 x_2(k) - a_3 x_3(k) + u(k)$$

$$x_2(k+1) = x_1(k)$$

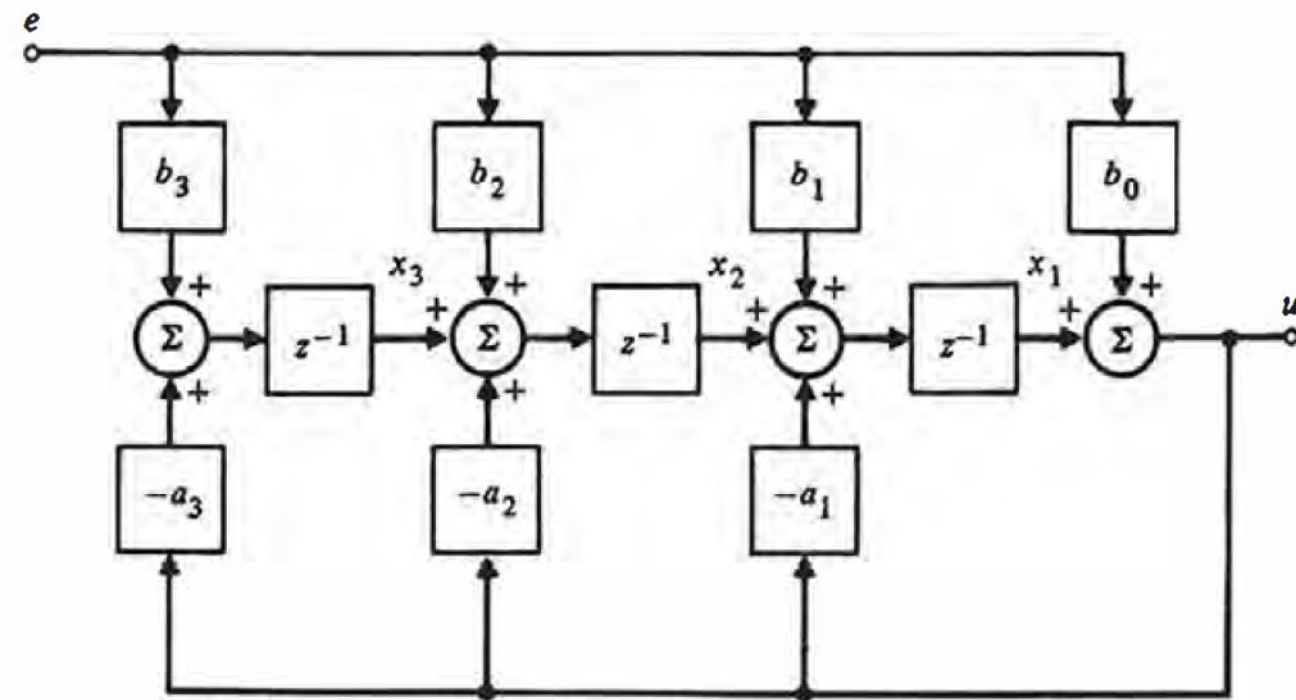
$$x_3(k+1) = x_2(k)$$

$$y(k) = b_1 x_1(k) + b_2 x_2(k) + b_3 x_3(k)$$

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k)$$

Forma canônica observável



$$\mathbf{x}(k+1) = \mathbf{A}_o \mathbf{x}(k) + \mathbf{B}_o e(k)$$

$$u(k) = \mathbf{C}_o \mathbf{x}(k) + \mathbf{D}_o e(k)$$

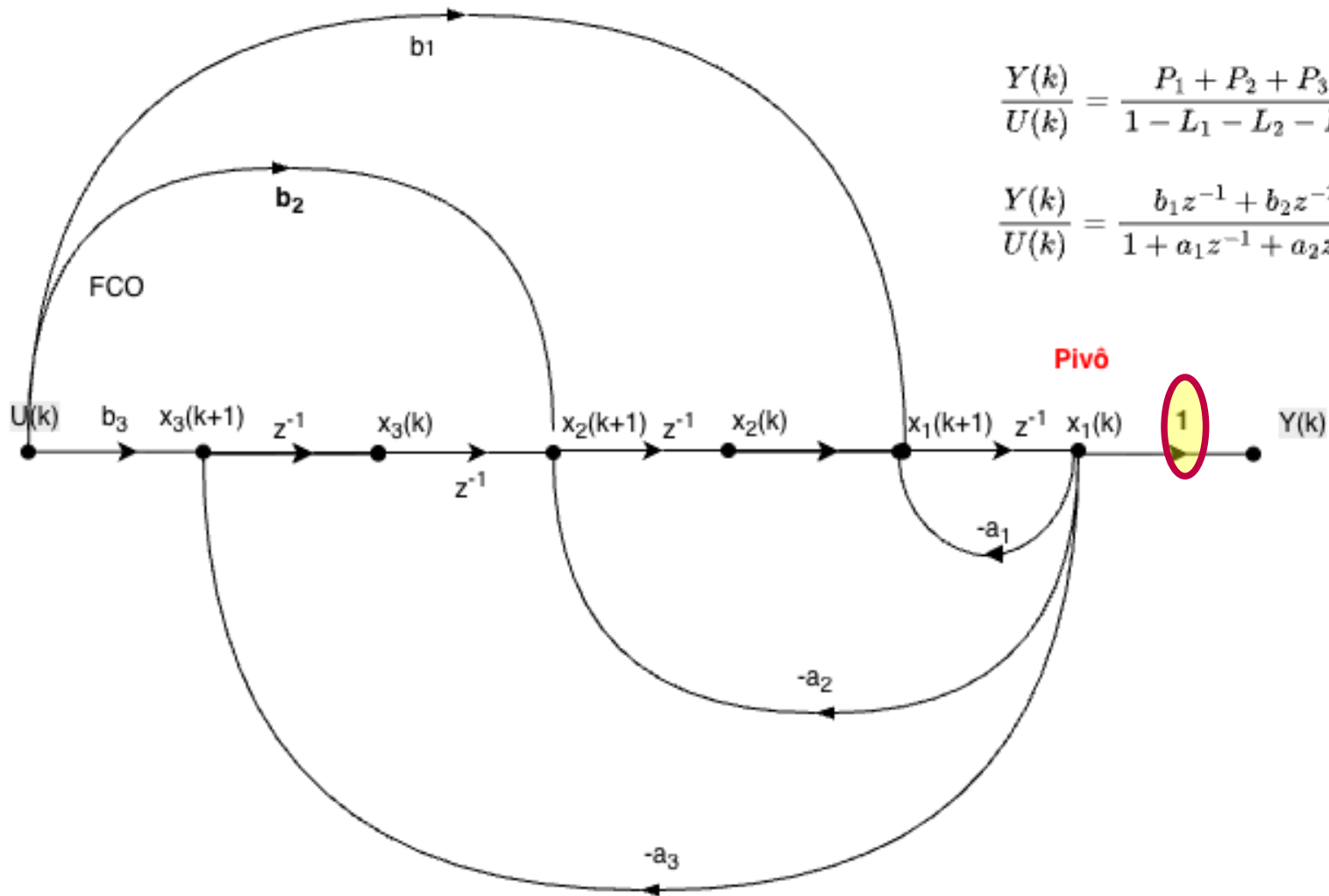
$$\mathbf{A}_o = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_o = \begin{bmatrix} b_1 - b_0 a_1 \\ b_2 - b_0 a_2 \\ b_3 - b_0 a_3 \end{bmatrix}$$

$$\mathbf{C}_o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_o = \begin{bmatrix} b_0 \end{bmatrix}$$

Forma canônica observável



$$\frac{Y(k)}{U(k)} = \frac{P_1 + P_2 + P_3}{1 - L_1 - L_2 - L_3}$$

$$\frac{Y(k)}{U(k)} = \frac{b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

Resposta ao Pulso Unitário

$$e_k = \begin{cases} 1, & (k = 0) \\ 0, & (k \neq 0) \end{cases} \\ \hat{=} \delta_k.$$

$$U(z) = H(z)$$

$$H(z) = \sum_{-\infty}^{\infty} u_k z^{-k} \hat{=} \sum_{-\infty}^{\infty} h_k z^{-k}$$

Transformada da resposta
ao pulso unitário

Exemplo: $U(z) = \frac{T}{2} \frac{1+z^{-1}}{1-z^{-1}} E(z)$

k	e_{k-1}	e_k	a_k	u_{k-1}	$u_k \equiv h_k$
0	0	1	$T/2$	0	$T/2$
1	1	0	$T/2$	$T/2$	T
2	0	0	0	T	T
3	0	0	0	T	T

$$\begin{aligned}
 H(z) &= \sum_{k=0}^{\infty} T z^{-k} - \frac{T}{2} \\
 &= \frac{T}{1-z^{-1}} - \frac{T}{2} \quad (1 < |z|) \\
 &= \frac{2T - T(1-z^{-1})}{2(1-z^{-1})} \\
 &= \frac{T + Tz^{-1}}{2(1-z^{-1})} \\
 &= \frac{T}{2} \frac{z+1}{z-1} \quad (1 < |z|).
 \end{aligned}$$

Convolução

Produto no domínio
da frequência:

$$U(z) = E(z) H(z)$$



$$u_k = \sum_{j=0}^k e_j h_{k-j}$$

Convolução no domínio
do tempo discreto

$$U(z) = \sum_{j=-\infty}^{\infty} e_j z^{-j} \sum_{l=-\infty}^{\infty} h_l z^{-l}$$

$$\sum_{k=-\infty}^{\infty} u_k z^{-k} = \sum_{k=-\infty}^{\infty} z^{-k} \sum_{j=-\infty}^{\infty} e_j h_{k-j}$$

$$u_0 = e_0 h_0$$

$$u_1 = e_0 h_1 + e_1 h_0$$

$$u_2 = e_0 h_2 + e_1 h_1 + e_2 h_0$$

$$u_3 = e_0 h_3 + e_1 h_2 + e_2 h_1 + e_3 h_0$$

BIBO Estabilidade

Entrada limitada: $|e_l| \leq M < \infty$ for all l

$$|u_k| \leq \left| \sum e_i h_{k-i} \right|$$

$$\leq \sum_{-\infty}^{\infty} |e_i| |h_{k-i}|$$

$$\leq M \sum_{-\infty}^{\infty} |h_{k-i}|$$

Saída limitada se:

$$\sum_{l=-\infty}^{\infty} |h_{k-l}| < \infty$$

Exemplo - BIBO Estabilidade

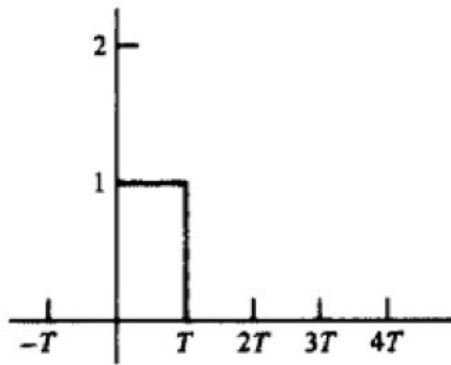
Sistema discreto: $u_k = a_1 u_{k-1} + b_0 e_k$

Resposta a um Pulso unitário: $u_0 = b_0, \quad u_1 = a_1 b_0, \quad u_2 = a_1^2 b_0, \quad \dots$
 $u_k = h_k = b_0 a^k, \quad k \geq 0.$

Estabilidade: $\sum_{l=-\infty}^{\infty} |h_l| = \sum_{l=0}^{\infty} b_0 |a_l|^k = b_0 \frac{1}{1 - |a|} \quad (|a| < 1)$
 $= \text{unbounded} \quad (|a| \geq 1)$

Modelo Discreto de Sistemas Amostrados

Pulso unitário



$$Y_1(s) = (1 - e^{-Ts}) G \frac{(s)}{s}$$

$$\begin{aligned} G(z) &= \mathcal{Z}\{Y_1(kT)\} \\ &= \mathcal{Z}\{\mathcal{L}^{-1}\{Y_1(s)\}\} \hat{=} \mathcal{Z}\{Y_1(s)\} \\ &= \mathcal{Z}\{(1 - e^{-Ts}) \frac{G(s)}{s}\}. \end{aligned}$$

$$\mathcal{Z}\{e^{-Ts} \frac{G(s)}{s}\} = z^{-1} \mathcal{Z}\{\frac{G(s)}{s}\}$$



$$G(z) = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{G(s)}{s} \right\}$$

Modelo discreto equivalente de $G(s)$

Exemplo: $G(s) = \frac{a}{s+a}$

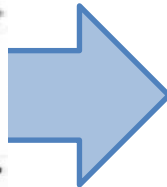
$$G(z) = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{G(s)}{s} \right\}$$

Modelo discreto equivalente com ZOH

$$\frac{G(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$\mathcal{L}^{-1} \left\{ \frac{G(s)}{s} \right\} = 1(t) - e^{-at} 1(t)$$

$$\begin{aligned} \mathcal{Z} \left\{ \frac{G(s)}{s} \right\} &= \frac{z}{z-1} - \frac{z}{z-e^{-aT}} \\ &= \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})} \end{aligned}$$

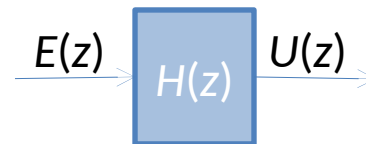


$$\begin{aligned} G(z) &= \frac{z-1}{z} \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})} \\ &= \underline{\underline{\frac{1-e^{-aT}}{z-e^{-aT}}}} \end{aligned}$$

4.4 Resposta Dinâmica

Procedimento:

1. $H(z)$
2. $E(z)$
3. $U(z) = E(z) H(z)$
4. $u(kT)$



Pulso Unitário

$$\begin{aligned}e_1(k) &= 1 & (k = 0) \\ &= 0 & (k \neq 0) \\ &= \delta_k;\end{aligned}$$

$$E_1(z) = \sum_{-\infty}^{\infty} \delta_k z^{-k} = z^0 = 1.$$

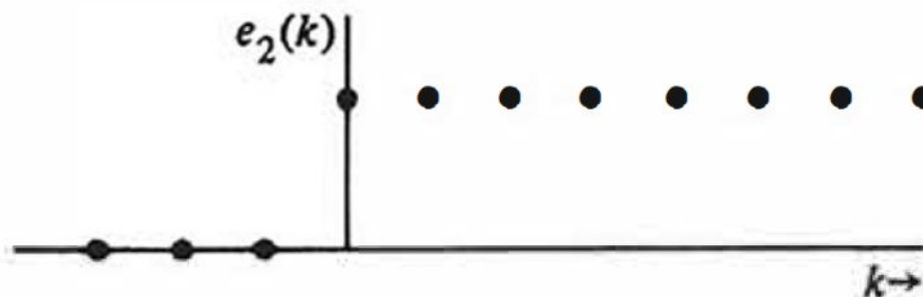
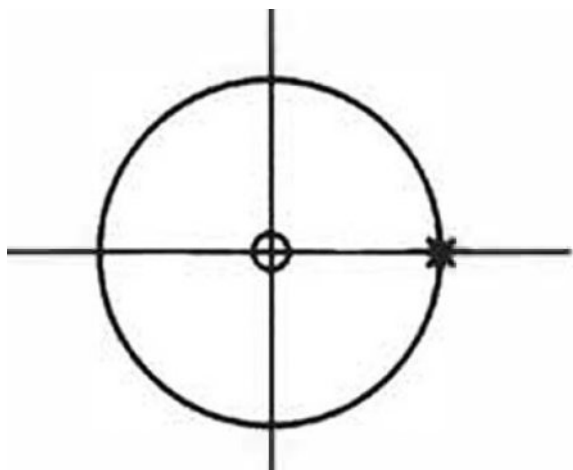
$$U(z) = E(z) H(z) \quad U(z) = H(z)$$

Função de transferência discreta é a resposta ao pulso unitário.

Degrau Unitário

$$\begin{aligned} e_2(k) &= 1 & (k \geq 0) \\ &= 0 & (k < 0) \\ &\triangleq 1(k). \end{aligned}$$

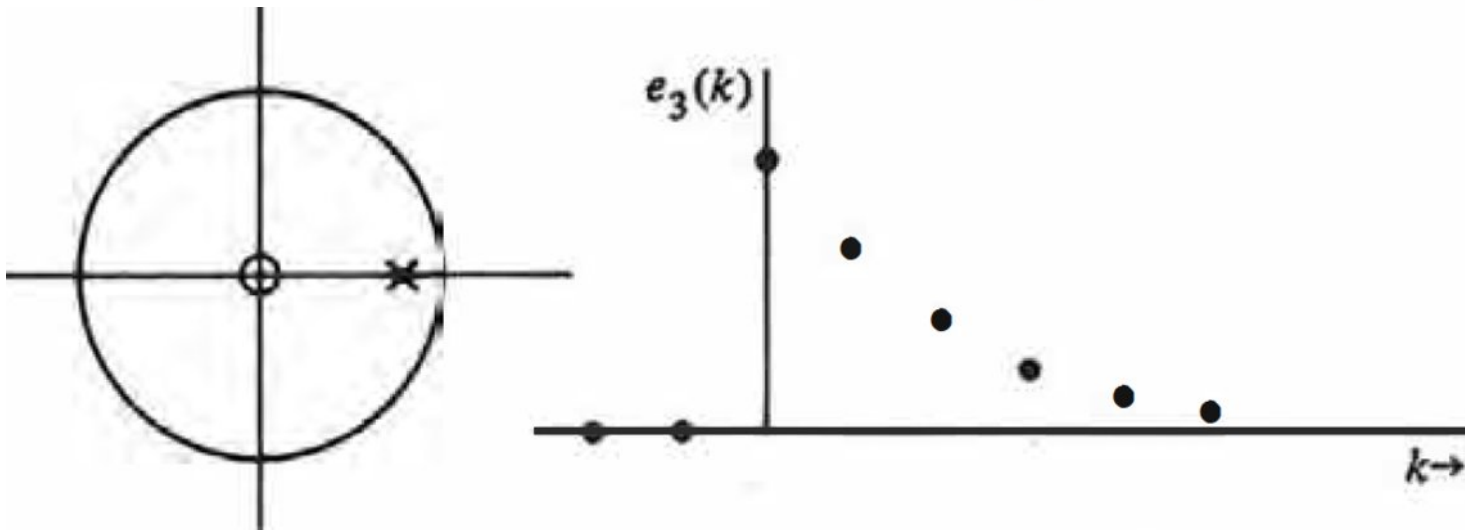
$$\begin{aligned} E_2(z) &= \sum_{k=-\infty}^{\infty} e_2(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} \\ &= \frac{1}{1 - z^{-1}} & (|z^{-1}| < 1) \\ &= \frac{z}{z - 1} & (|z| > 1). \end{aligned}$$



Exponencial

$$e_3(k) = r^k \quad (k \geq 0) \\ = 0 \quad (k < 0)$$

$$\begin{aligned} E_3(z) &= \sum_{k=0}^{\infty} r^k z^{-k} \\ &= \sum_{k=0}^{\infty} (r z^{-1})^k \\ &= \frac{1}{1 - r z^{-1}} \quad (|r z^{-1}| < 1) \\ &= \frac{z}{z - r} \quad (|z| > |r|). \end{aligned}$$



Senóide Genérica $e_4(k) = r^k \left(\frac{e^{jk\theta} + e^{-jk\theta}}{2} \right) 1(k)$

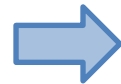
Decompondo em parcelas:

$$e_5(k) = r^k e^{jk\theta} 1(k)$$

$$E_5(z) = \sum_{k=0}^{\infty} r^k e^{jk\theta} z^{-k}$$

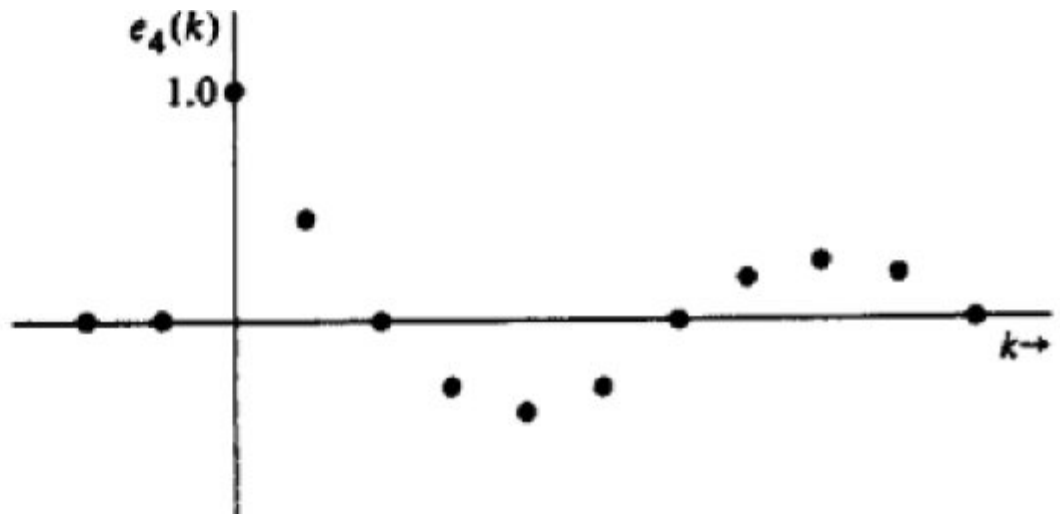
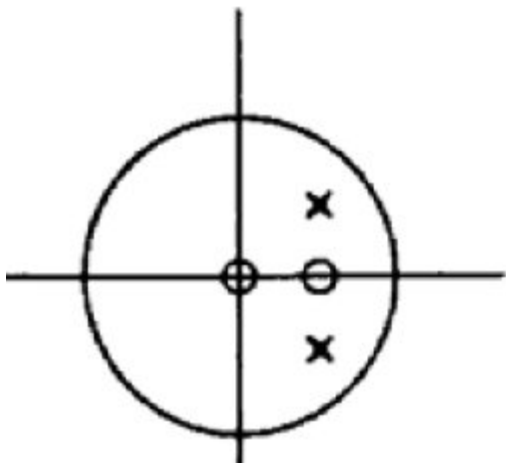
$$= \sum_{k=0}^{\infty} (r e^{j\theta} z^{-1})^k = \frac{1}{1 - r e^{j\theta} z^{-1}}$$

$$\mathcal{Z}\{r^k e^{-j\theta k} 1(k)\} = \frac{z}{z - r e^{-j\theta}} \quad (|z| > r)$$



$$E_4(z) = \frac{1}{2} \left\{ \frac{z}{z - r e^{j\theta}} + \frac{z}{z - r e^{-j\theta}} \right\}$$

$$= \frac{z(z - r \cos \theta)}{z^2 - 2r(\cos \theta)z + r^2} \quad (|z| > r)$$



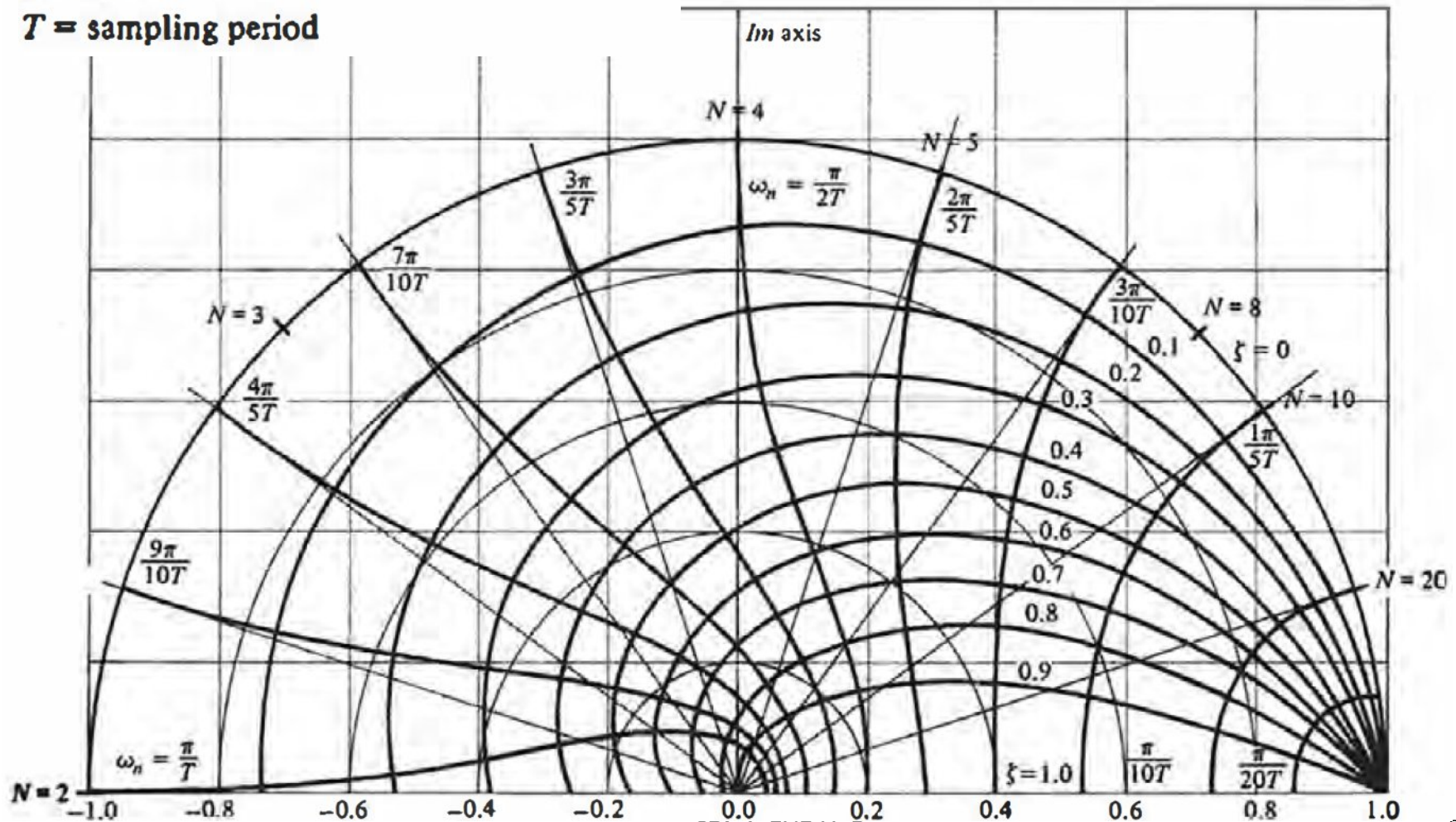
Mapeamento de pólos - ζ e ω_n ctes

z = plane loci of roots of constant ζ and ω_n

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$z = e^{Ts}$$

T = sampling period



4.4.5 Correspondência com sinais contínuos

$$y(t) = e^{-at} \cos bt \, 1(t)$$

$$aT = 0.3567,$$

$$bT = \pi/4.$$

Com $T=1$ seg.

$$y(kT) = (e^{-0.3567})^k \cos \frac{\pi k}{4} 1(k)$$

$$= (0.7)^k \cos \frac{\pi k}{4} 1(k)$$

$$= e_4(k).$$

$$s_{1,2} = -a + jb, -a - jb.$$

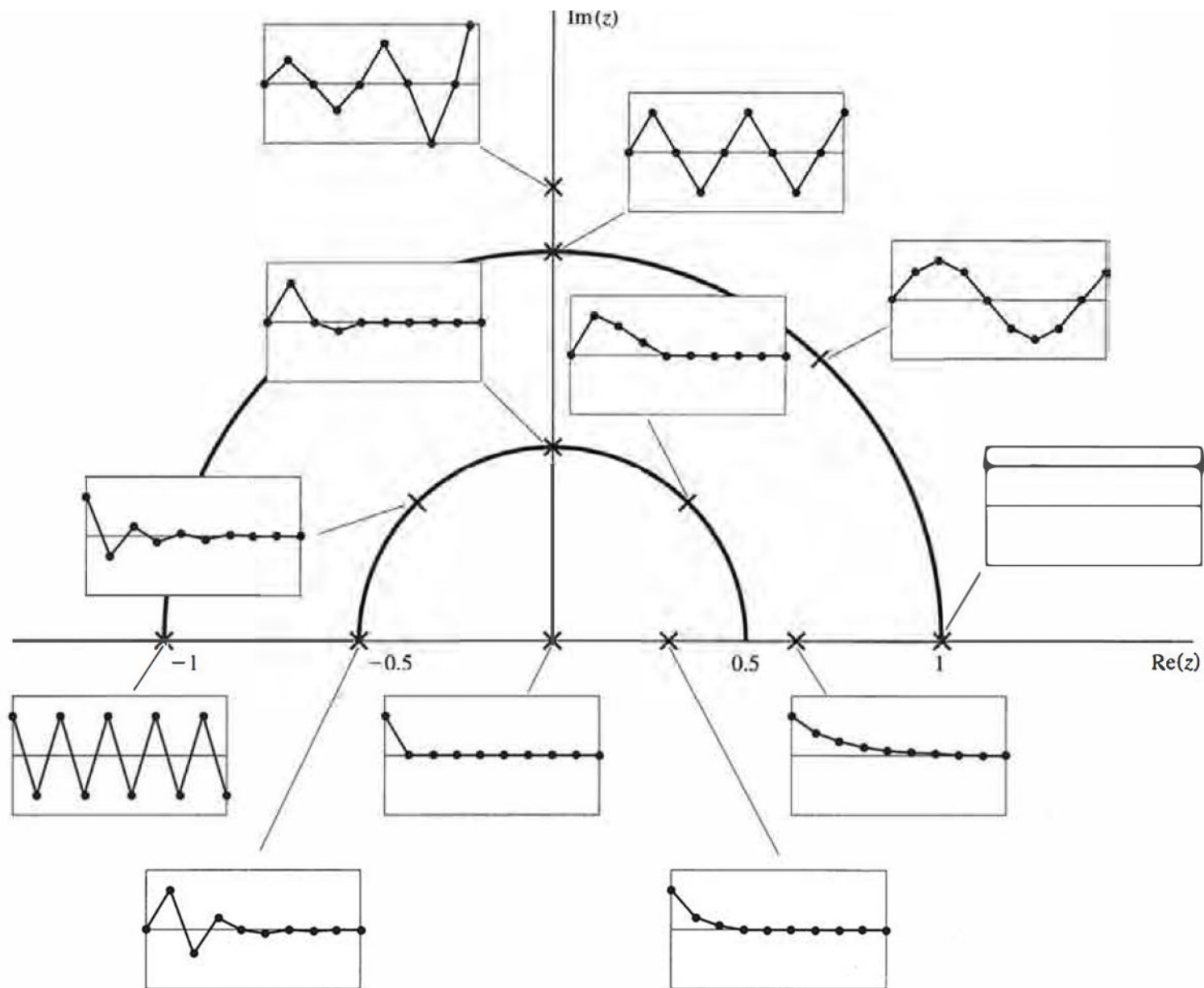
$$z_{1,2} = re^{j\theta}, re^{-j\theta}$$

$$r = e^{-aT}, \quad \theta = bT$$

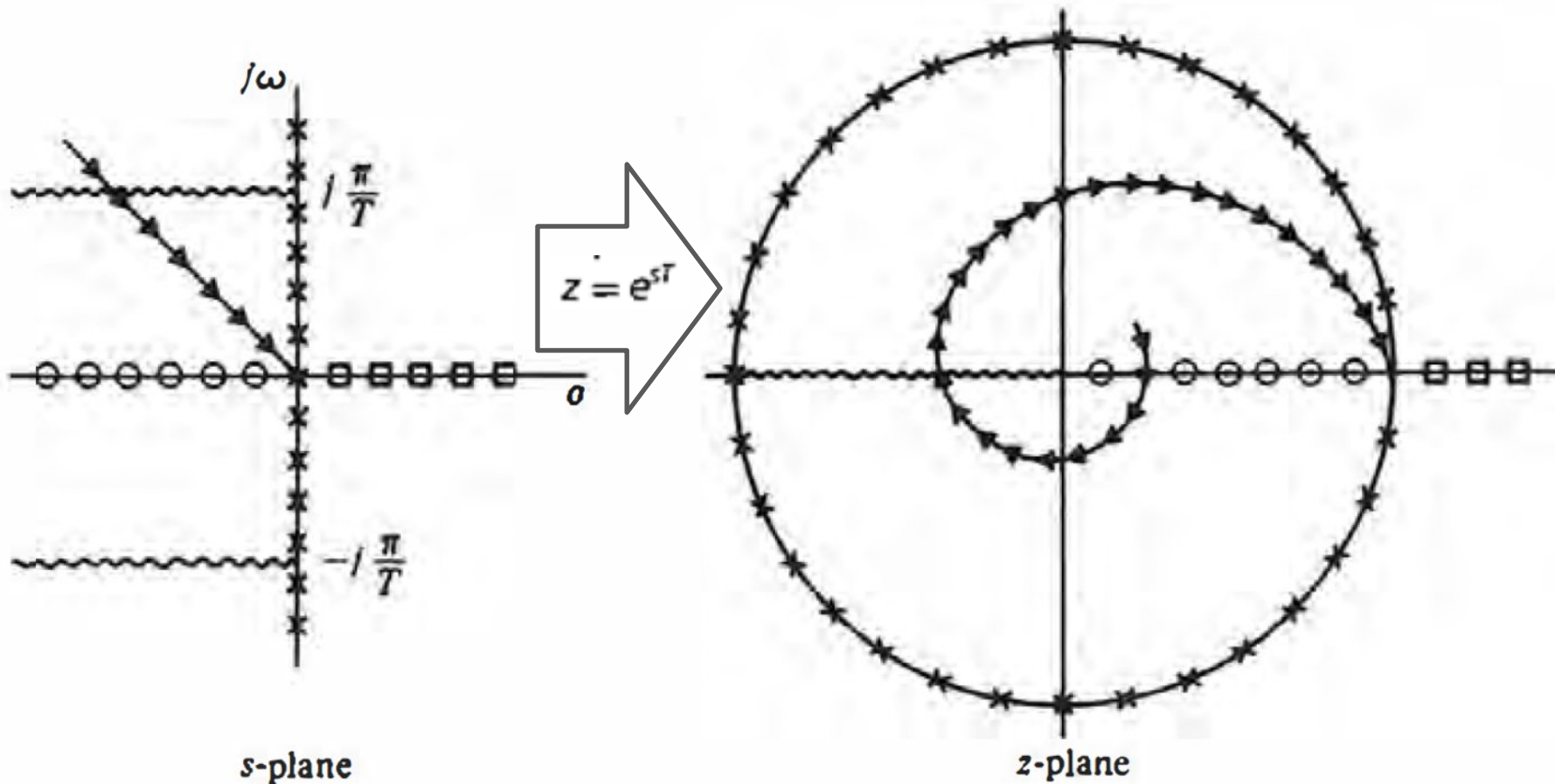
$$z_{1,2} = e^{s_1 T}, e^{s_2 T}.$$

$$z = e^{sT}$$

Sequências
temporais
associadas
às posições
dos pólos
no plano Z



Mapeamento de Linhas Características



Mapeamento não unívoco: $s_2 = s_1 + j\frac{2\pi}{T}N$

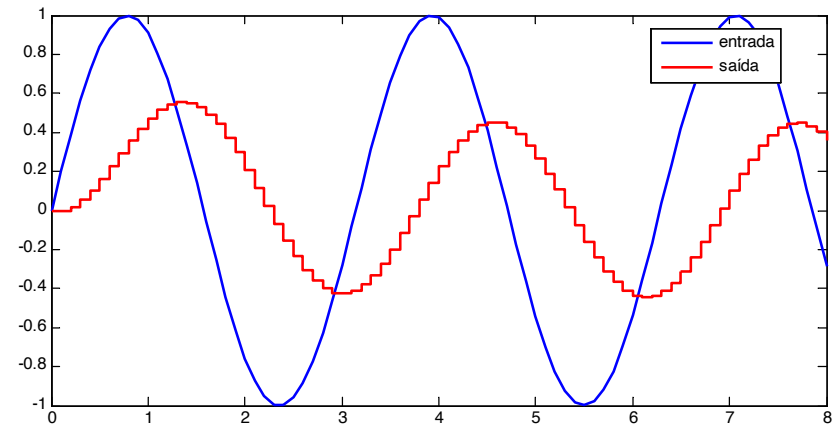
4.5 Resposta em Frequência

$$E(z) = \frac{1}{2} \left\{ \frac{z}{z - e^{+j\omega_o T}} + \frac{z}{z - e^{-j\omega_o T}} \right\}$$

$$U(z) = \frac{1}{2} \left\{ \frac{zH(z)}{z - e^{+j\omega_o T}} + \frac{zH(z)}{z - e^{-j\omega_o T}} \right\}$$

$$U_{ss}(z) = \frac{1}{2} \frac{H(e^{j\omega_o T})z}{z - e^{j\omega_o T}} + \frac{1}{2} \frac{H(e^{-j\omega_o T})z}{z - e^{-j\omega_o T}}$$

$$U(z) = E(z) H(z)$$



Considerando $H(e^{j\omega_o T}) = A(\omega_o T)e^{j\psi(\omega_o T)}$

$$U_{ss}(z) = \frac{A}{2} \frac{e^{j\psi} z}{z - e^{-j\omega_o T}} + \frac{A}{2} \frac{e^{-j\psi} z}{z - e^{-j\omega_o T}} \quad \Rightarrow \quad U_{ss}(kT) = \frac{A}{2} e^{j\psi} e^{j\omega_o T k} + \frac{A}{2} e^{-j\psi} e^{-j\omega_o T k} = A \cos(\omega_o T k + \psi),$$

$$A = |H(e^{j\omega_o T})| \quad \psi = \angle(H(e^{j\omega_o T}))$$

4.6 Propriedades da Transformada Z

1. Linearidade

$$\begin{aligned}\mathcal{Z}\{\alpha f_1(kT) + \beta f_2(kT)\} &= \sum_{k=-\infty}^{\infty} \{\alpha f_1(k) + \beta f_2(k)\} z^{-k} \\ &= \alpha \mathcal{Z}\{f_1(k)\} + \beta \mathcal{Z}\{f_2(k)\} \\ &= \alpha F_1(z) + \beta F_2(z).\end{aligned}$$

1. Convolução de sequências temporais

$$\mathcal{Z}\left\{\sum_{l=-\infty}^{\infty} f_1(l) f_2(k-l)\right\} = F_1(z) F_2(z)$$

Propriedades da Transformada Z...

3. Deslocamento no tempo $\mathcal{Z}\{f(k+n)\} = z^{+n} F(z)$

4. Escalonamento em z $\mathcal{Z}\{r^{-k} f(k)\} = F(rz)$

Ex:
$$\mathcal{Z}\{1(k)\} = \sum_{k=0}^{\infty} z^{-k} = \frac{z}{z-1}$$

$$\mathcal{Z}\{r^{-k} 1(k)\} = \frac{rz}{rz-1} = \frac{z}{z-(1/r)}$$

Propriedades da Transformada Z...

5. Teorema do Valor Final $\lim_{k \rightarrow \infty} f(k) = \lim_{z \rightarrow 1} (z - 1) F(z)$

6. a) Inversão por divisão longa

$$\text{Ex: } H(z) = \frac{T}{2} \frac{z+1}{z-1}, \quad |z| > 1, \quad E_3(z) = \frac{z}{z-0.5}, \quad |z| > 0.5.$$

$$\begin{aligned} U(z) &= E_3(z) H(z) \\ &= \frac{z}{z-0.5} \frac{T}{2} \frac{z+1}{z-1}, \quad |z| > 1 \end{aligned} \quad U(z) = \frac{T}{2} \frac{1+z^{-1}}{1-1.5z^{-1}+0.5z^{-2}},$$

Propriedades da Transformada Z...

6.a) Inversão por divisão longa Ex:...

$$\begin{array}{r}
 \frac{T}{2}[1 + 2.5z^{-1} + 3.25z^{-2} + 3.625z^{-3} + \dots] \\
 1 - 1.5z^{-1} + 0.5z^{-2} \quad) \quad 1 + z^{-1} \\
 \underline{1 - 1.5z^{-1} + 0.5z^{-2}} \\
 2.5z^{-1} - 0.5z^{-2} \\
 \underline{2.5z^{-1} - 3.75z^{-2} + 1.25z^{-3}} \\
 3.25z^{-2} - 1.25z^{-3} \\
 \underline{3.25z^{-2} - 4.875z^{-3} + 1.625z^{-4}} \\
 3.625z^{-3} - 1.625z^{-4} \\
 \underline{3.625z^{-3} - \dots}
 \end{array}$$

Por comparação direta:

$$U(z) = \sum_0^{\infty} u(k)z^{-k},$$

$$u_0 = T/2,$$

$$u_1 = (T/2)2.5,$$

$$u_2 = (T/2)3.25$$

⋮

Propriedades da Transformada Z...

6.b) Inversão por expansão em frações parciais

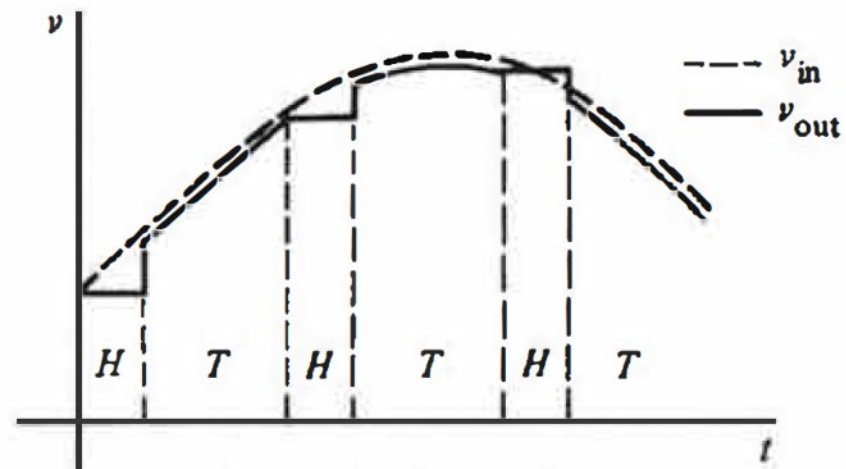
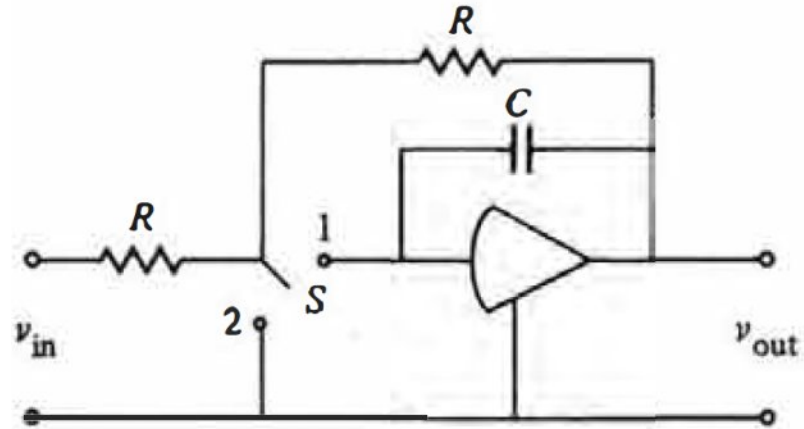
$$\text{Ex: } U(z) = \frac{T}{2} \frac{1+z^{-1}}{1-z^{-1}} \frac{1}{1-0.5z^{-1}} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.5z^{-1}}$$

$$A = \frac{T}{2} \frac{2}{0.5} = 2T \quad B = \frac{T}{2} \frac{1+2}{1-2} = -\frac{3T}{2}$$

$$\begin{aligned} u_k &= Ae_2(k) + Be_3(k) \\ &= 2Te_2(k) - \frac{3T}{2}e_3(k) \\ &= \left(2T - \frac{3T}{2} \left(\frac{1}{2} \right)^k \right) 1(k) \\ &= \frac{T}{2} \left[4 - \frac{3}{2^k} \right] 1(k). \end{aligned}$$

5-Sistemas Amostrados

1. Sample & Hold



H = Hold; S in position 2.
 T = Track; S in position 1.

S&H – Análise Matemática

Separar Sample de Hold

$$\text{---} \underset{T}{\nearrow} \text{---} \quad r^*(t) = \sum_{k=-\infty}^{\infty} r(t) \delta(t - kT)$$

Amostragem => modulação por impulsos

Propriedade* $\int_{-\infty}^{\infty} f(t) \delta(t - a) dt = f(a)$

Transf. de Laplace de um sinal amostrado

Para o impulso

$$\int_{-\infty}^t \delta(\tau) d\tau = 1(t)$$

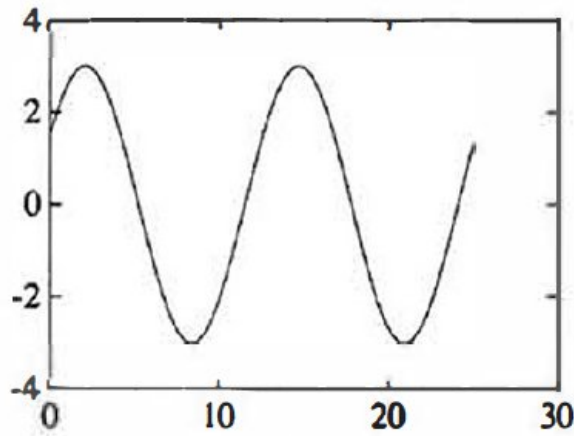
$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(\tau) e^{-s\tau} d\tau = 1$$

$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r^*(\tau) e^{-s\tau} d\tau$$

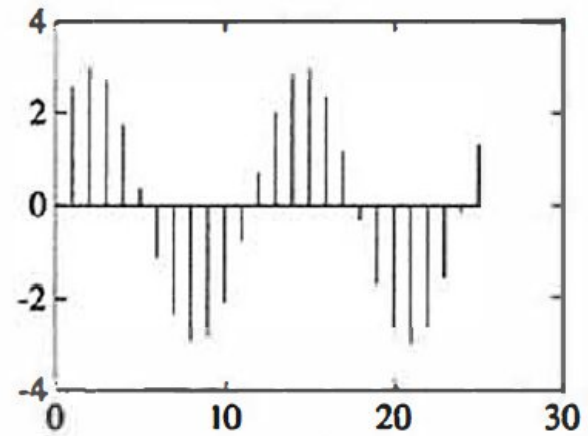
$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} r(\tau) \delta(\tau - kT) e^{-s\tau} d\tau$$

$$R^*(s) = \sum_{k=-\infty}^{\infty} r(kT) e^{-skT}$$

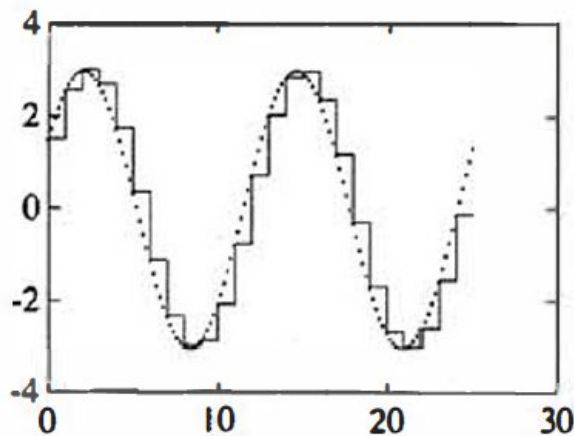
Segurador de Ordem Zero



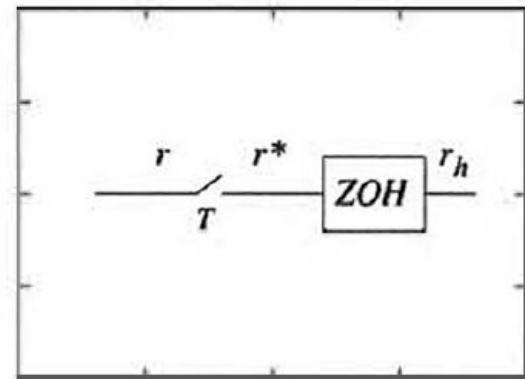
(a)



(b)



(c)



(d)

Segurador de Ordem Zero

$$r_h(t) = r(kT) \quad kT \leq t < kT + T.$$

$$p(t) = 1(t) - 1(t - T)$$

$$\begin{aligned} ZOH(s) &= \mathcal{L}\{p(t)\} \\ &= \int_0^{\infty} [1(t) - 1(t - T)]e^{-st} dt \\ &= (1 - e^{-sT})/s. \end{aligned}$$

5.2 Espectro de um Sinal Amostrado

Representação da série de impulsos por Fourier

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{n=-\infty}^{\infty} C_n e^{j(2\pi n/T)t}$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t - kT) e^{-jn(2\pi t/T)} dt$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn(2\pi t/T)} dt \quad C_n = \frac{1}{T}$$

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j(2\pi n/T)t}$$

Espectro de um Sinal Amostrado...

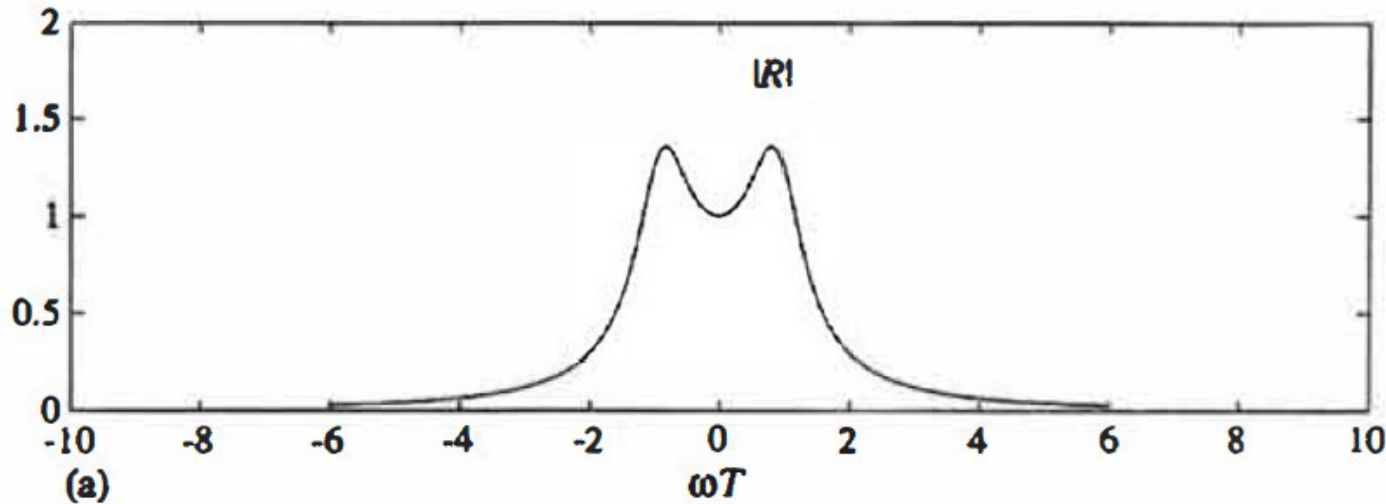
$$\omega_s = \frac{2\pi}{T} \quad \text{Freq. De amostragem (rad/s)}$$

$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r(t) \left\{ \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \right\} e^{-st} dt$$

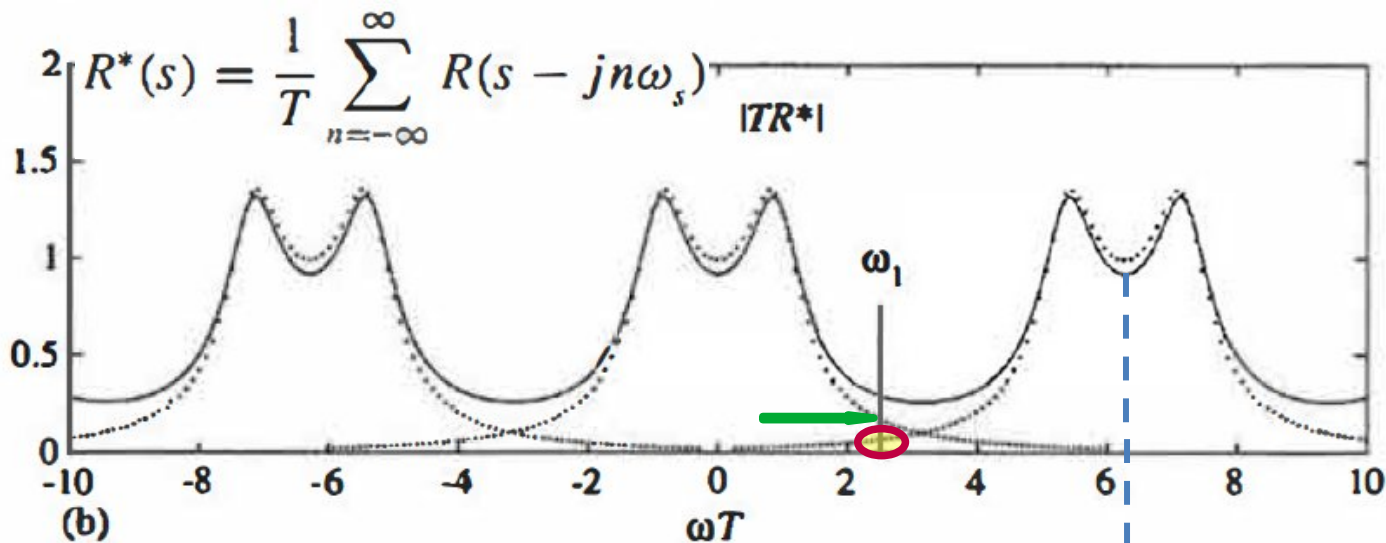
$$R^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} r(t) e^{jn\omega_s t} e^{-st} dt$$

$$R^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} r(t) e^{-(s-jn\omega_s)t} dt \quad \Rightarrow \quad R^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} R(s - jn\omega_s)$$

Espectro de um Sinal Amostrado...



Espectro de Amplitude

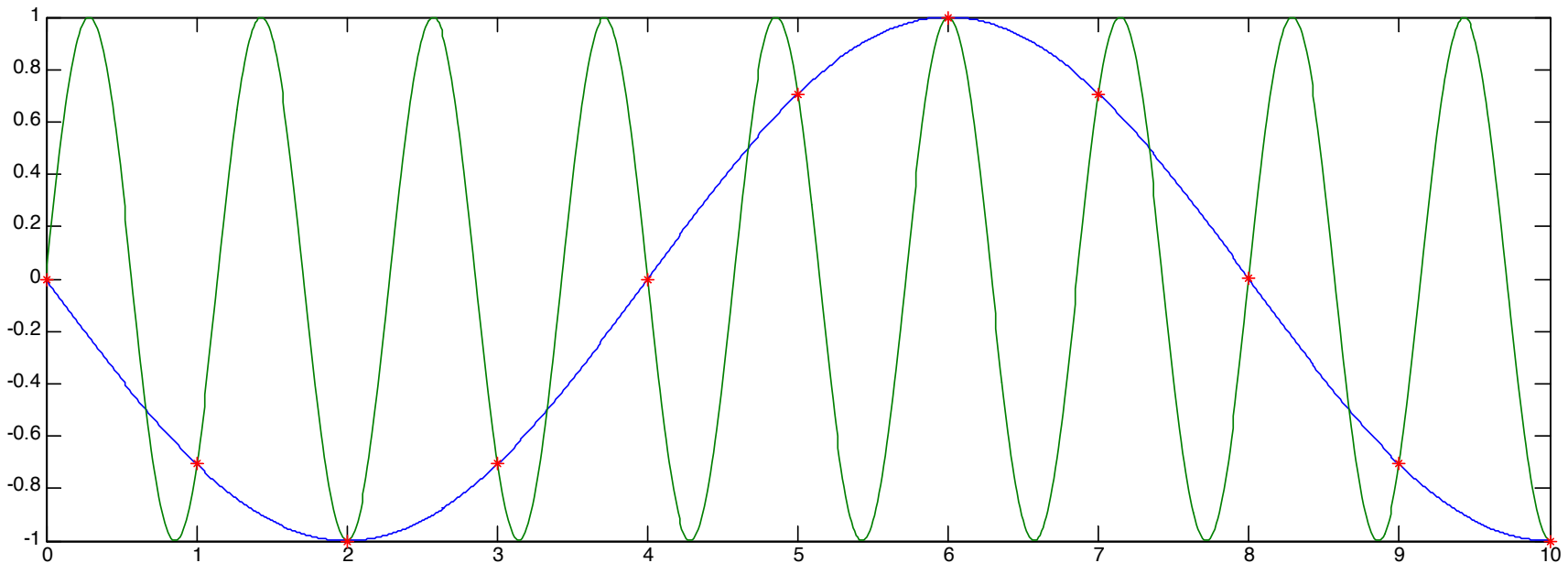


Espectro do sinal Amostrado
→ tem “aliasing”

$R(j\omega_1)$ é alias de
 $R(j\omega_1 - \frac{2\pi}{T})$

Exemplo – Alias $\omega_1 + n \frac{2\pi}{T} = \omega_1 + n\omega_s$

Sinais de 1/8 Hz e 7/8Hz amostrados a 1Hz

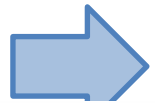


$f_0 = 1/8 - 1 = -7/8 \rightarrow$ senoide invertida

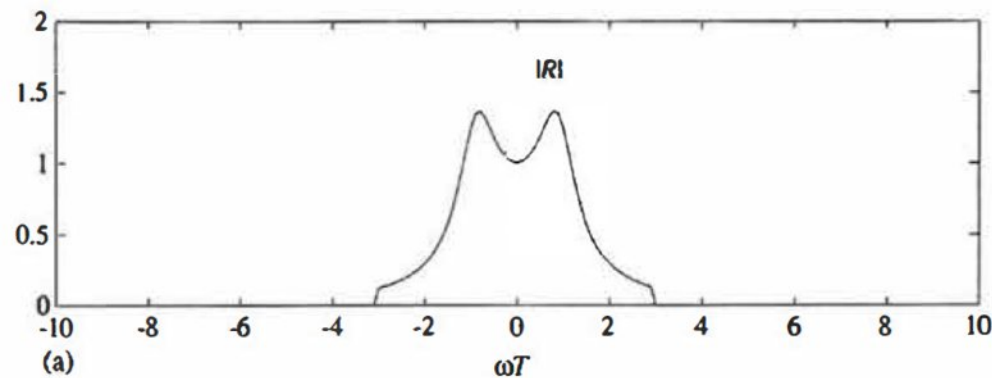
```
w1=2*pi*1/8; w2=2*pi*7/8; t=0:0.01:10; T=0:10,plot(t,sin(-w1*t),t,sin(w2*t),T,sin(-w1*T),'*')
```

Alias

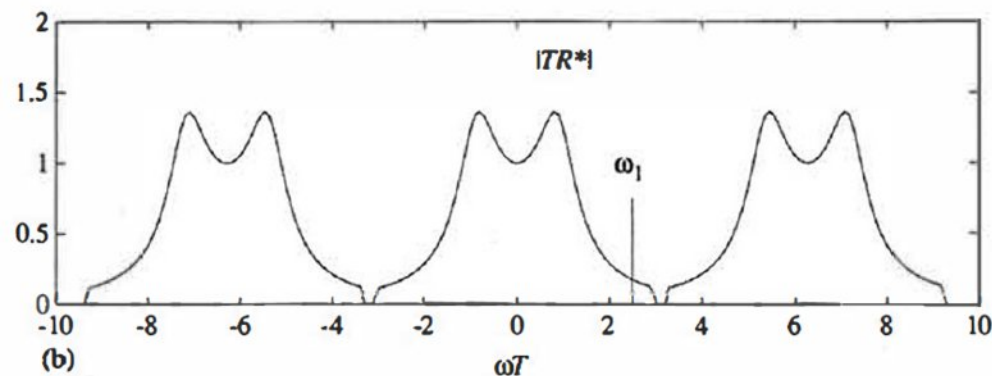
Componentes de alta frequência geram erros quando amostrados



preceder S&H por filtro passa-baixas anti-alias



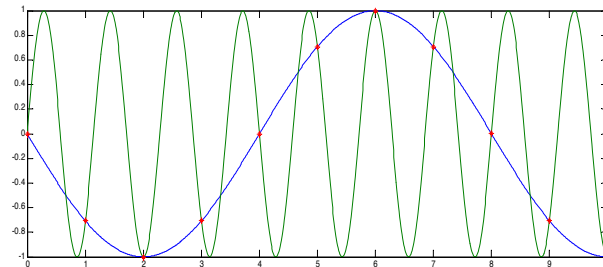
Remover componentes espectrais acima de π/T



$\pi/T \rightarrow$ frequência de Nyquist

Teorema da Amostragem

- $\pi/T \rightarrow$ frequência de Nyquist



Sinais sem componentes acima da freq. de Nyquist
são representados/recuperados de forma única
de suas amostras

A frequência de amostragem, $\omega_s = 2\pi/T$ deve ser pelo menos duas vezes a maior frequência (π/T) do sinal.

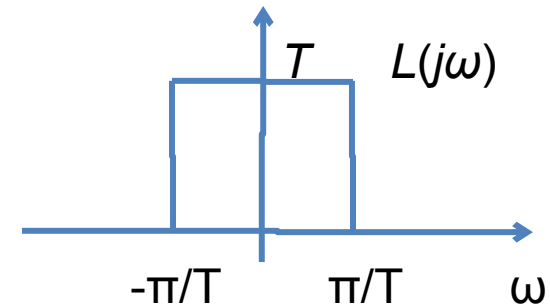
Interpolação de Sinais Amostrados

- Filtro Passa-Baixas Ideal

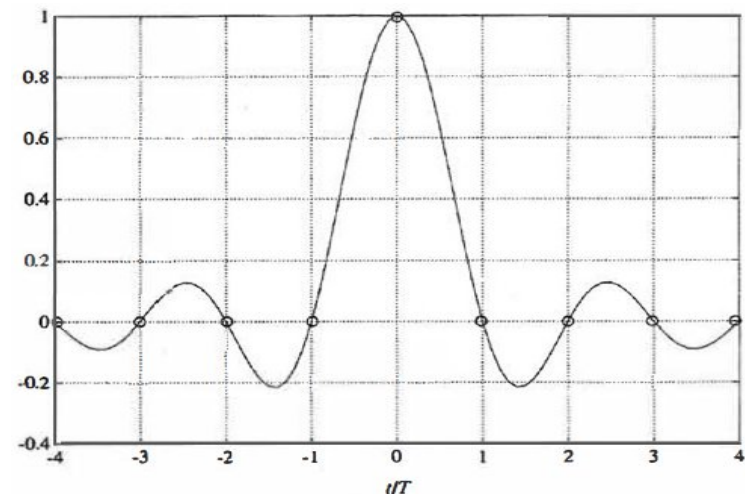
→ reconstrução de $r(t)$ a partir de suas amostras

$$R(j\omega) = L(j\omega)R^*(j\omega)$$

$$\begin{aligned}\ell(t) &= \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} T e^{j\omega t} d\omega \\ &= \frac{T}{2\pi} \frac{e^{j\omega t}}{jt} \bigg|_{-\pi/T}^{\pi/T} \\ &= \frac{T}{2\pi jt} (e^{j(\pi t/T)} - e^{-j(\pi t/T)}) \\ &= \frac{\sin(\pi t/T)}{\pi t/T} \\ &\triangleq \text{sinc} \frac{\pi t}{T}.\end{aligned}$$



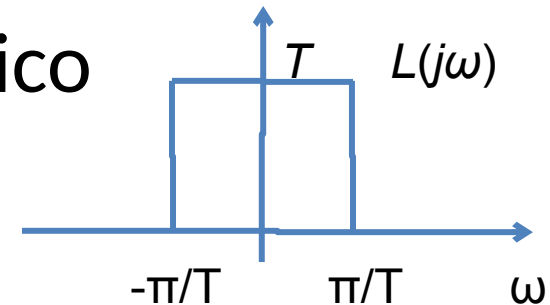
Resposta ao Impulso **acausal**



Interpolação de Sinais Amostrados ...

- Recuperação do sinal analógico

$$r(t) = \int_{-\infty}^{\infty} r(\tau) \sum_{k=-\infty}^{\infty} \delta(\tau - kT) \operatorname{sinc} \frac{\pi(t - \tau)}{T} d\tau$$



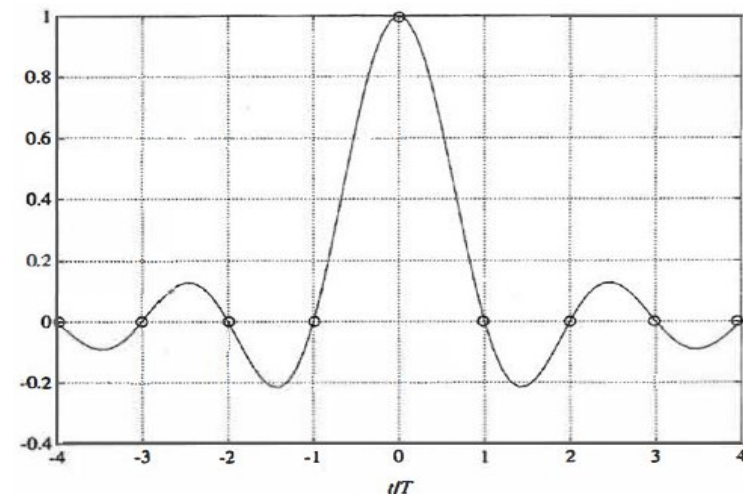
Propriedade de amostragem

$$r(t) = \sum_{k=-\infty}^{\infty} r(kT) \operatorname{sinc} \frac{\pi(t - kT)}{T}$$

Filtro ideal:

Acausal $\rightarrow$ Não é realizável!!

$\rightarrow$ adicionar atraso(ZOH)



Recuperação do sinal analógico, via ZOH do conversor A/D

$$ZOH(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega}$$

(Aceitar um atraso de $T/2$)

$$ZOH(j\omega) = e^{-j\omega T/2} \left\{ \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right\} \frac{2j}{j\omega}$$

$$ZOH(j\omega) = T e^{-j\omega T/2} \frac{\sin(\omega T/2)}{\omega T/2}$$

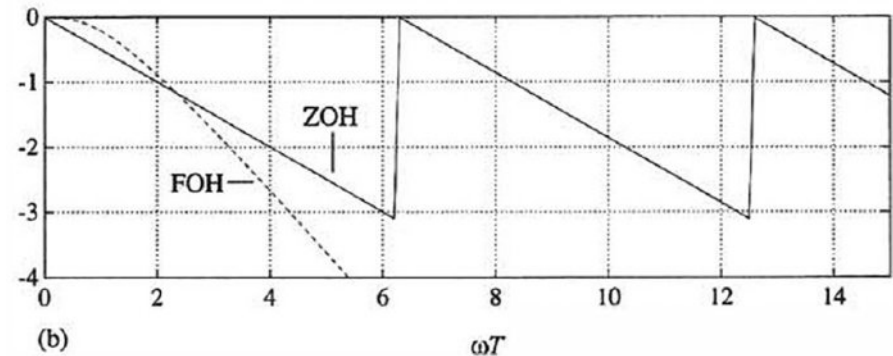
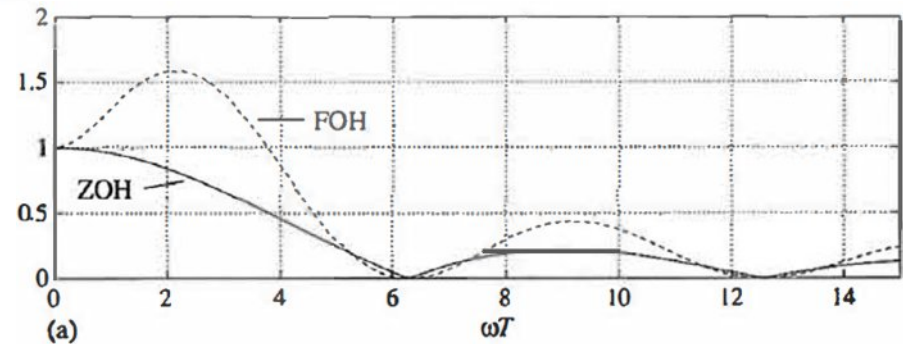
$$ZOH(j\omega) = e^{-j\omega T/2} T \text{sinc}(\omega T/2)$$



Atraso $T/2$

$$|ZOH(j\omega)| = T \left| \text{sinc} \frac{\omega T}{2} \right|$$

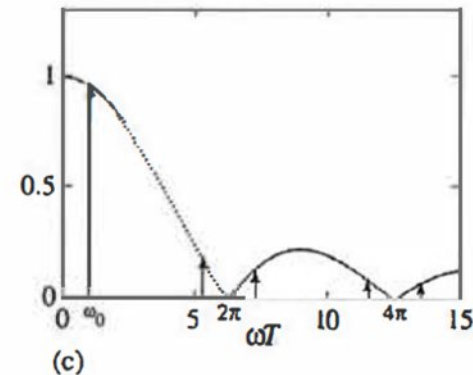
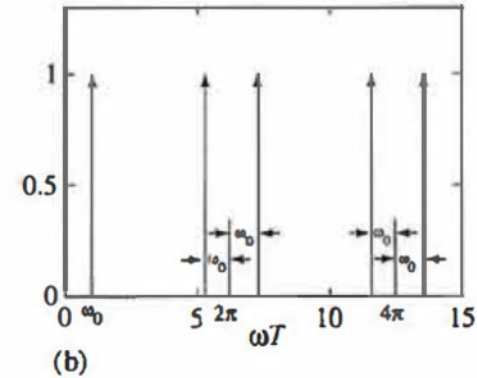
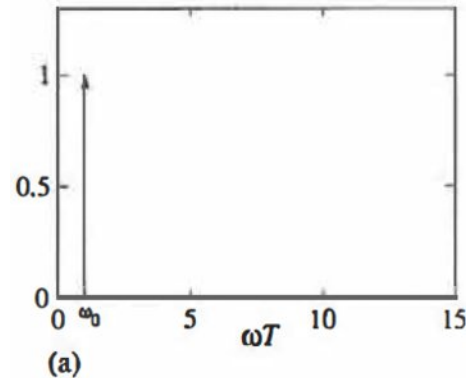
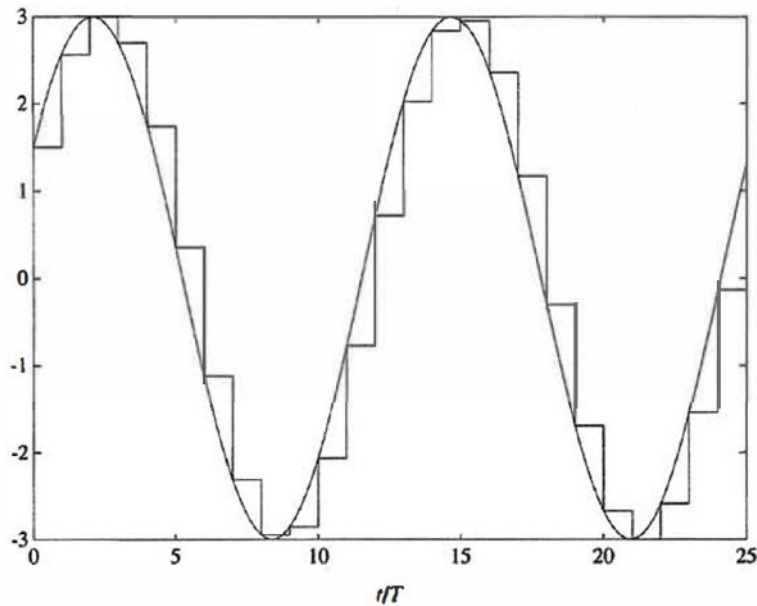
$$\angle ZOH(j\omega) = \frac{-\omega T}{2}$$



Senóide amostrada

$$R(j\omega) = \pi A[e^{j\phi}\delta(\omega - \omega_o) + e^{-j\phi}\delta(\omega + \omega_o)]$$

$$ZOH(j\omega) = T e^{-j\omega T/2} \text{sinc}(\omega T/2)$$



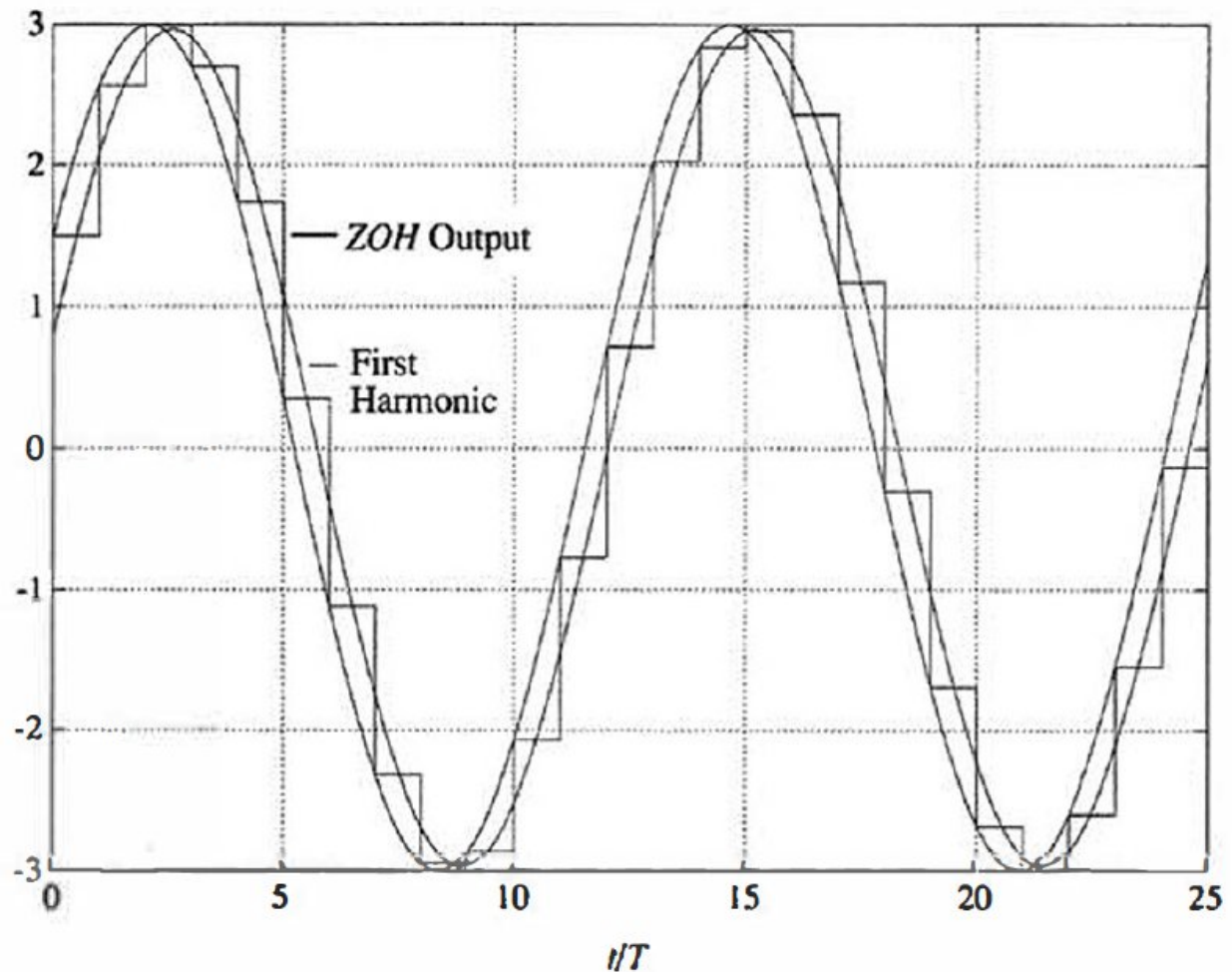
$$v(t) = A \sin(\omega_0 t) \xrightarrow{\text{ZOH}}$$

Harmônica Fundamental:

$$v_1(t) = A[\text{sinc}(\omega T/2)] \sin[\omega_o(t - \frac{T}{2})]$$

Senóide amostrada

$$ZOH(j\omega) = T e^{-j\omega T/2} \text{sinc}(\omega T/2)$$



$v(t) = A \sin(\omega_0 t)$  Harmônica Fundamental:

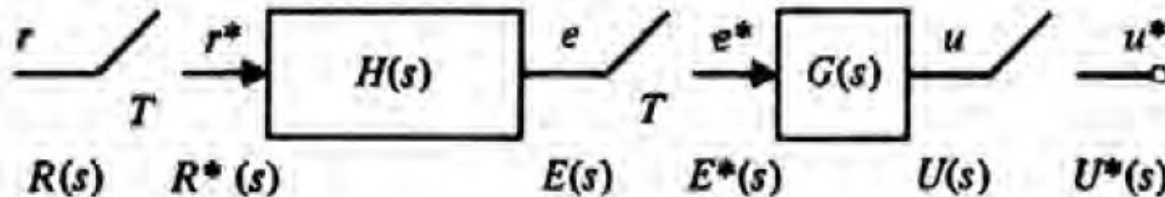
$$v_1(t) = A [\text{sinc}(\omega T/2)] \sin[\omega_0(t - \frac{T}{2})]$$

5.4 Análise de Diagramas de Blocos

Funções de transferência discretas equivalentes?

$$E(s) = R^*(s)H(s),$$

$$U(s) = E^*(s)G(s).$$

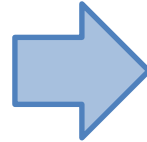


Definição: $U(z) = U^*(s) \mid_{e^{sT}=z}$
 $z = e^{sT}$

Transformada de Laplace
do sinal amostrado

Propriedade da Amostragem $(E(s)*G(s))^*=E(s)*G(s)^*$

$$U^*(s) = (E^*(s)G(s))^*$$



Propriedade básica para
Simplificar diagramas de blocos

$$U^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} E^*(s - jn\omega_s) G(s - jn\omega_s)$$

$$E^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} E(s - jk\omega_s) \quad E^*(s - jn\omega_s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} E(s - jk\omega_s - jn\omega_s).$$

$$E^*(s - jn\omega_s) = \frac{1}{T} \sum_{\ell=-\infty}^{\infty} E(s - j\ell\omega_s) \quad \Rightarrow \quad U^*(s) = E^*(s) \frac{1}{T} \sum_{n=-\infty}^{\infty} G(s - jn\omega_s) \\ = E^*(s). \quad = E^*(s)G^*(s). \quad \text{QED}$$

Importante :

$$U(s) = E(s)G(s)$$

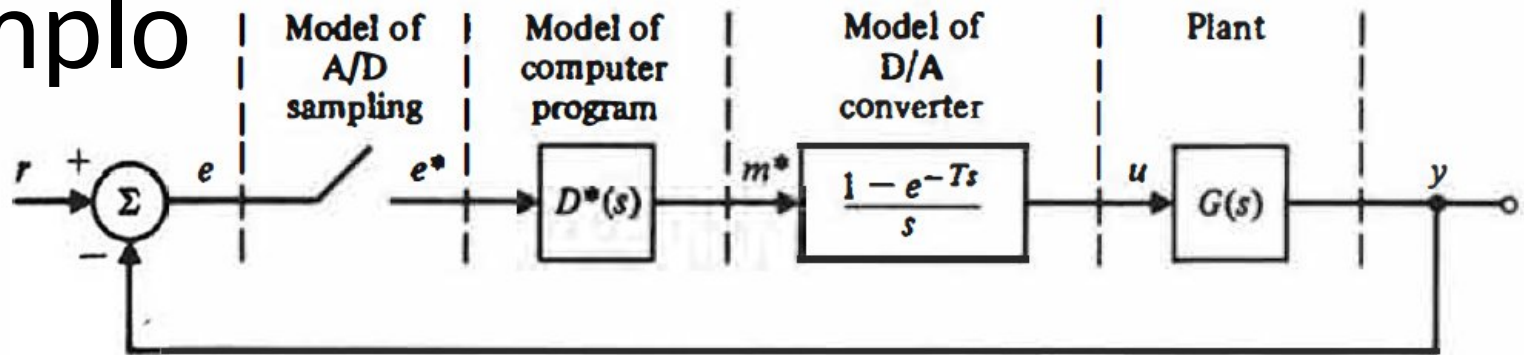
$$U^*(s) = (E(s)G(s))^*$$

$$U^*(s) \neq E^*(s)G(s)^*$$

notação

$$U(z) = U^*(s) \Big|_{e^{sT}=z}$$

Exemplo



$$E(s) = R - Y,$$

$$M^*(s) = E^* D^*,$$

$$U = M^* \left[\frac{1 - e^{-Ts}}{s} \right]$$

$$Y = GU.$$

$$E^* = R^* - Y^*,$$

$$M^* = E^* D^*,$$

$$U^* = M^*,$$

$$Y^* = [GU]^*.$$

$$Y^* = \left[GM^* \left(\frac{1 - e^{-Ts}}{s} \right) \right]^*$$

Retirando os sinais periódicos:

$$Y^* = (1 - e^{-Ts}) M^* \left(\frac{G}{s} \right)^*$$

$$Y^* = (1 - e^{-Ts}) E^* D^* (G/s)^*$$

$$Y^* = (1 - e^{-Ts}) D^* (G/s)^* [R^* - Y^*]$$

Denominando

$$(1 - e^{-Ts}) D^* (G/s)^* = H^*$$

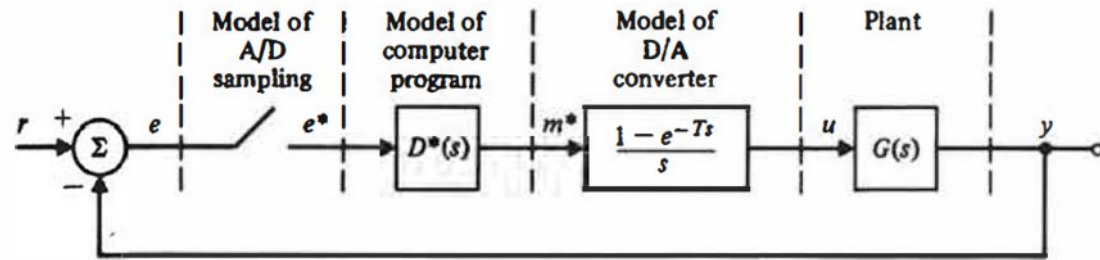


$$Y^* = \frac{H^*}{1 + H^*} R^*$$

Exemplo

$$G(s) = \frac{a}{s+a}$$

$$e^{-aT} = \frac{1}{2}$$



Controlador: $u(kT) = u(kT - T) + K_0 e(kT)$,

Solução: $D(z) = \frac{U(z)}{E(z)} = \frac{K_0}{1 - z^{-1}} = \frac{K_0 z}{z - 1} \Rightarrow D^*(s) = \frac{K_0 e^{sT}}{e^{sT} - 1}$

Modelo equivalente
(Transf. Inv. Ao degrau)

$$(1 - e^{-Ts})(G(s)/s)^* = (1 - e^{-Ts}) \left(\frac{a}{s(s+a)} \right)^*$$

$$= (1 - e^{-Ts}) \left(\frac{1}{s} - \frac{1}{s+a} \right)^*$$

$$(1 - e^{-Ts})(G(s)/s)^* = (1 - e^{-Ts}) \left(\frac{1}{1 - e^{-Ts}} - \frac{1}{1 - e^{-aT} e^{-Ts}} \right)$$

$$(1 - e^{-Ts})(G(s)/s)^* = \frac{(1/2)e^{-Ts}}{1 - (1/2)e^{-Ts}}$$

$$= \frac{1/2}{e^{Ts} - 1/2}$$

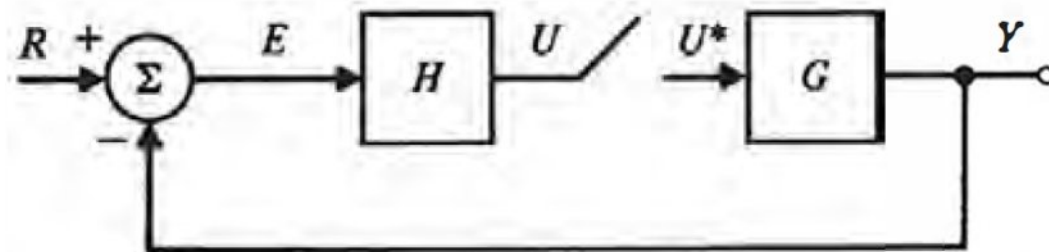
$$H^*(s) = \frac{K_0}{2} \frac{e^{sT}}{(e^{sT} - 1)(e^{sT} - 1/2)}$$

$$Y(s) = R^* \frac{D^*}{1 + H^*} \frac{(1 - e^{-Ts})}{s} G(s)$$

Outro exemplo

Eq. do sistema:

$$\begin{cases} E = R - Y \\ U = HE \\ Y = U^* G \end{cases}$$



Amostrando:

$$\begin{cases} E^* = R^* - Y^*, \\ U^* = (HE)^*, \\ Y^* = U^* G^*. \end{cases}$$

$$U^* = \frac{(HR)^*}{1 + (HG)^*} \quad Y^* = \frac{(HR)^*}{1 + (HG)^*} G^*$$



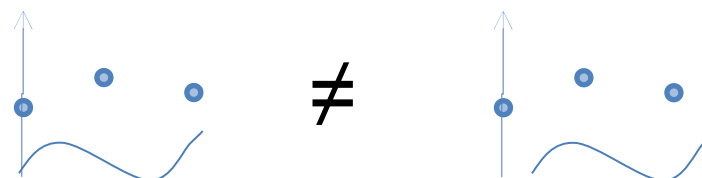
Não existe
função de transferência (Y^*/R^*)

$$\begin{aligned} U^* &= (H(R - Y))^* \\ &= (HR)^* - (HY)^*. \end{aligned}$$

$$U^* = (HR)^* - (HU^*G)^*$$

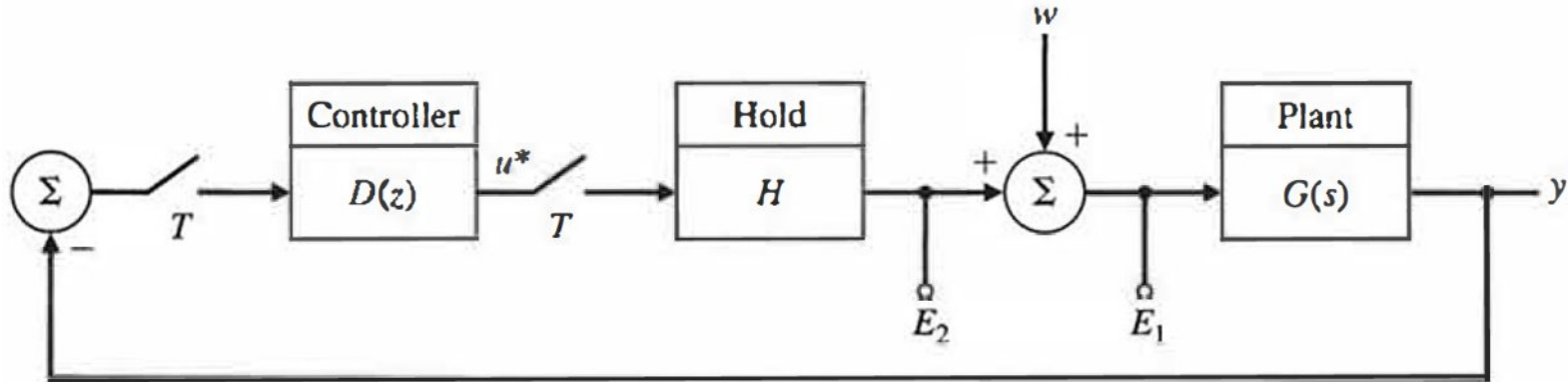
$$U^* = (HR)^* - U^*(HG)^*$$

Sistema variante no tempo!!



Obtenção Experimental do Ganho de Malha Aberta: sinal w , $r = 0$

1) Sinais w genéricos



$$E_1 = W + U^* H$$

$$E_2 = U^* H$$

$$Y = WG + U^* HG$$

$$Y^* = (WG)^* + U^* (HG)^*$$

$$U^* = -D^* Y^*$$

$$U^*(s) = -\frac{D^*(WG)^*}{1 + D^*(HG)^*}$$

$$E_1 = W - \frac{D^*(WG)^*}{1 + D^*(HG)^*} H$$

$$E_2 = -\frac{D^*(WG)^*}{1 + D^*(HG)^*} H$$



função de transferência MA (E_2^*/E_1^*)

(sinal W fica misturado)

2) Sinal senoidal (uma exponencial por vez) $w = 2\pi\delta(\omega - \omega_0)$

$$(GW)^* = \frac{1}{T} \sum_{k=-\infty}^{k=\infty} G(j\omega - jk \frac{2\pi}{T}) 2\pi\delta(\omega - \omega_0 - \frac{2\pi k}{T})$$

$$(GW)^* = \frac{1}{T} G(j\omega_0) 2\pi\delta(\omega - \omega_0) \quad \omega_0 < \pi/T \rightarrow \text{sem sobreposição do espectro}$$

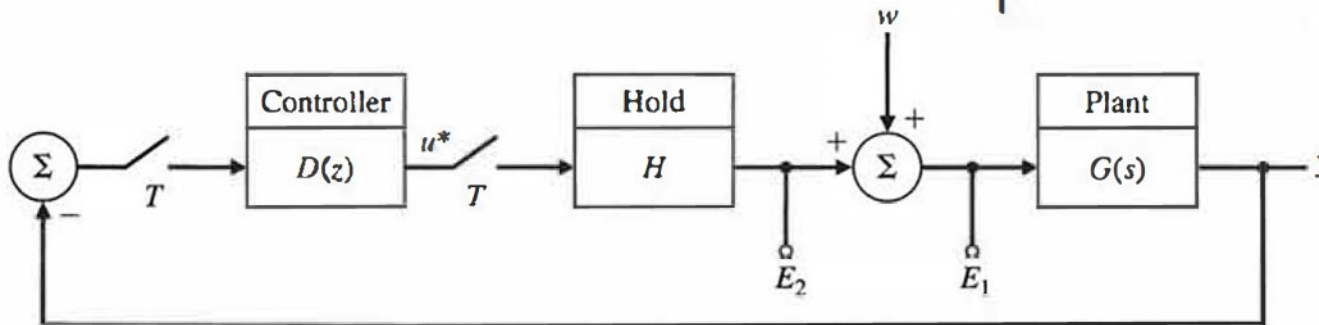
$$= \frac{1}{T} GW|_{\omega_0}$$

$$\frac{E_2}{E_1} = - \frac{\frac{1}{T} D^* GH}{1 + D^* (GH)^* - \frac{1}{T} D^* GH}$$

$$|G| = 0 \text{ for } |\omega| > \pi/T$$

$$G^* = G \quad \text{p/ } |\omega| < \pi/T$$

$$\frac{E_2}{E_1} = -D^* (GH)^* \quad \exists \text{ F.T. MA!}$$



3) F.T. MA: G com filtro passa-baixas anti-alias

$$|G| = 0 \quad \text{p/ } |\omega| > \pi/T$$

4) w aplicado via Sample & Hold

$$E_1 = U^* H + W^* H$$

$$E_2 = U^* H$$

$$U^* = -D^* Y^*$$

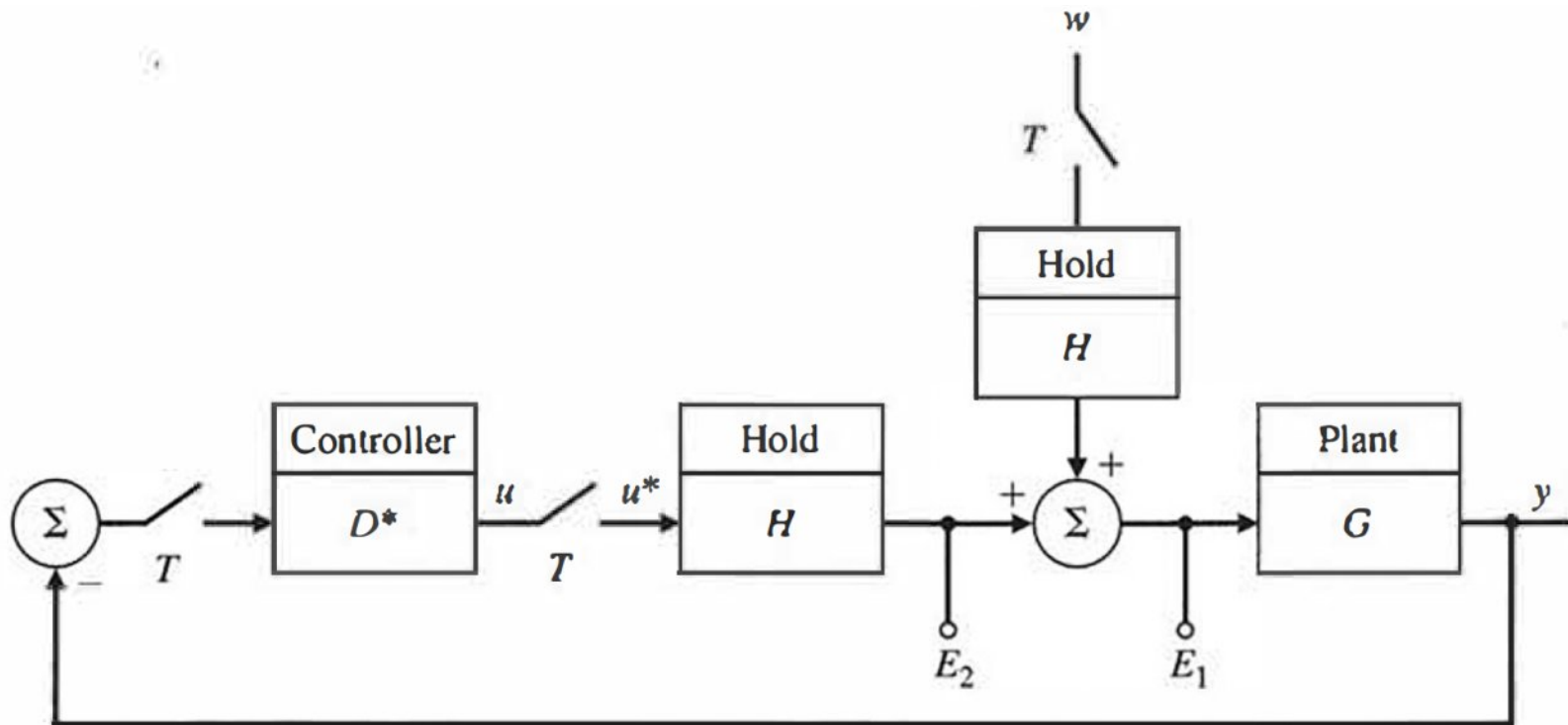
$$Y = U^* H G + W^* H G$$

$$E_1 = \frac{W^* H}{1 + D^* (H G)^*}$$

$$E_2 = -\frac{D^* (H G)^*}{1 + D^* (H G)^*} W^* H$$

$$\frac{E_2}{E_1} = -D^* (H G)^*$$

F.T. MA discreta



Transformada Z modificada

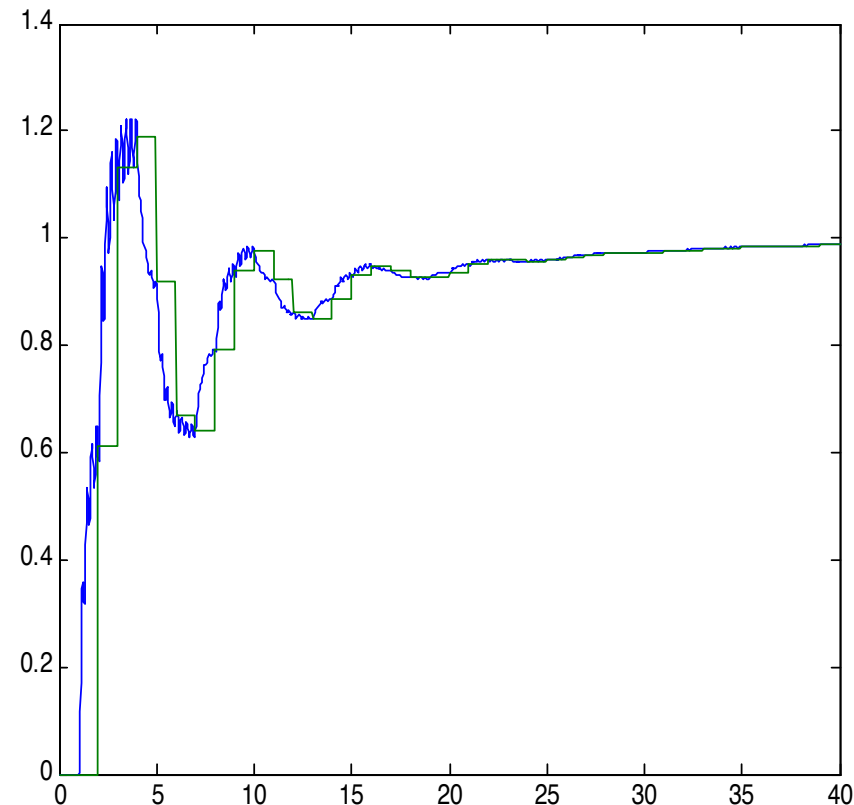
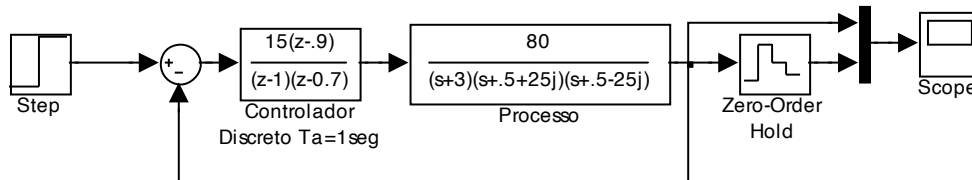
$$G(z, m) = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{G(s)}{s} e^{mTs} \right\} \quad 0 \leq m < 1$$

Permite calcular valores entre os instantes de amostragem

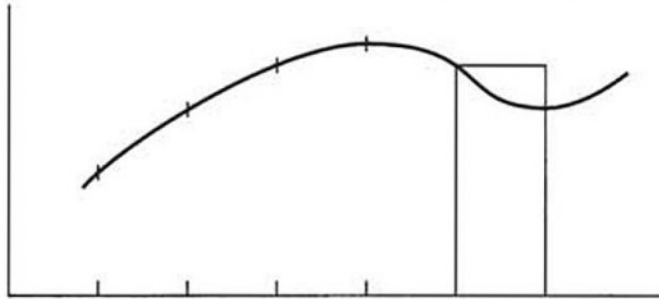
Verificar oscilações escondidas

→ escolher outra taxa de amostragem!!

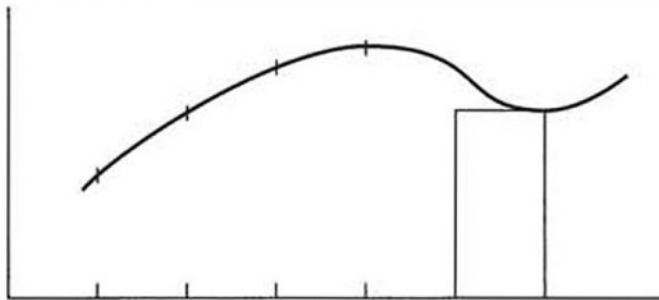
→ Simulink



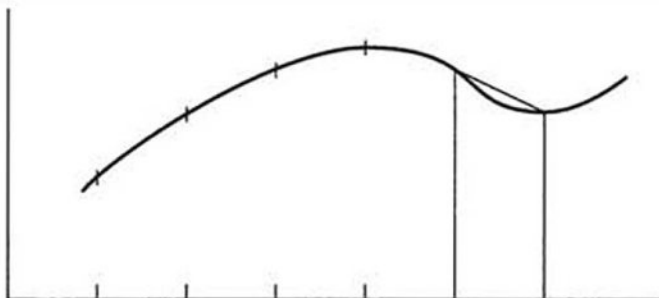
6. Sistemas Discretos Equivalentes



(a) Forward Rectangular Rule



(b) Backward Rectangular Rule



(c) Trapezoid Rule

Exemplo:

$$\frac{U(s)}{E(s)} = H(s) = \frac{a}{s + a}$$

$$\dot{u} + au = ae \quad (\text{Eq. Diferencial})$$

$$u(t) = \int_0^t [-au(\tau) + ae(\tau)] d\tau,$$

$$u(kT) = \int_0^{kT-T} [-au + ae] d\tau + \int_{kT-T}^{kT} [-au + ae] d\tau$$

$$= u(kT - T) + \left\{ \begin{array}{l} \text{area of } -au + ae \\ \text{over } kT - T \leq \tau < kT. \end{array} \right.$$

$$u(kT) = u(kT-T) + \text{Área}$$

Alguns métodos de integração simples

1) Regra Retangular em Avanço (Euler)

$$\begin{aligned}u_1(kT) &= u_1(kT - T) + T[-au_1(kT - T) + ae(kT - T)] \\&= (1 - aT)u_1(kT - T) + aTe(kT - T).\end{aligned}$$

$$\begin{aligned}H_F(z) &= \frac{aTz^{-1}}{1 - (1 - aT)z^{-1}} \\&= \frac{a}{(z - 1)/T + a}\end{aligned}$$

1) Regra Retangular em Atraso

$$\begin{aligned}u_2(kT) &= u_2(kT - T) + T[-au_2(kT) + ae(kT)] \\&= \frac{u_2(kT - T)}{1 + aT} + \frac{aT}{1 + aT}e(kT).\end{aligned}$$

$$\begin{aligned}H_B(z) &= \frac{aT}{1 + aT} \frac{1}{1 - z^{-1}/(1 + aT)} = \frac{aTz}{z(1 + aT) - 1} \\&= \frac{a}{(z - 1)/Tz + a} \quad (\text{backward rectangular rule})\end{aligned}$$

3) Regra Trapezoidal

$$\begin{aligned}
 u_3(kT) &= u_3(kT - T) + \frac{T}{2}[-au_3(kT - T) \\
 &\quad + ae(kT - T) - au_3(kT) + ae(kT)] \\
 &= \frac{1 - (aT/2)}{1 + (aT/2)}u_3(kT - T) + \frac{aT/2}{1 + (aT/2)}[e_3(kT - T) + e_3(kT)].
 \end{aligned}$$

$$\begin{aligned}
 H_T(z) &= \frac{aT(z + 1)}{(2 + aT)z + aT - 2} \\
 &= \frac{a}{(2/T)[(z - 1)/(z + 1)] + a} \quad \text{(trapezoid rule)}
 \end{aligned}$$

Alguns métodos de discretização

Method

Approximation

Forward rule	$s \leftarrow \frac{z-1}{T}$	$z = 1 + Ts,$
--------------	------------------------------	---------------

Backward Rule	$s \leftarrow \frac{z-1}{Tz}$	$z = \frac{1}{1-Ts},$
---------------	-------------------------------	-----------------------

Trapezoid Rule	$s \leftarrow \frac{2}{T} \frac{z-1}{z+1}$	$z = \frac{1+Ts/2}{1-Ts/2}$
----------------	--	-----------------------------

$H(s)$

Method

Transfer function

$$\frac{a}{s+a}$$

Forward rule

$$H_F = \frac{a}{(z-1)/T + a}$$

$$\frac{a}{s+a}$$

Backward rule

$$H_B = \frac{a}{(z-1)/Tz + a}$$

$$H_T(z) = H(s)|_{s=\frac{2}{T} \frac{z-1}{z+1}}$$

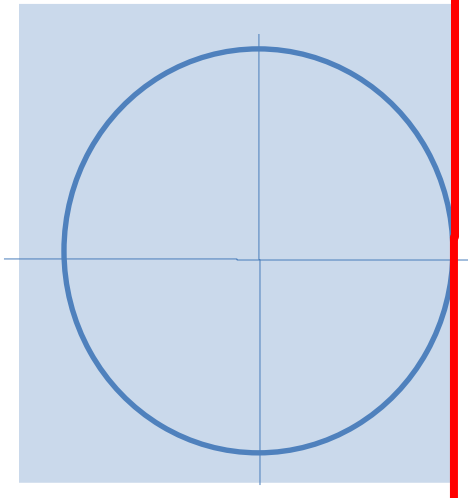
$$\frac{a}{s+a}$$

Trapezoid rule

$$H = \frac{a}{(2/T)\{(z-1)/(z+1)\} + a}$$

Mapeamento da Região de Estabilidade

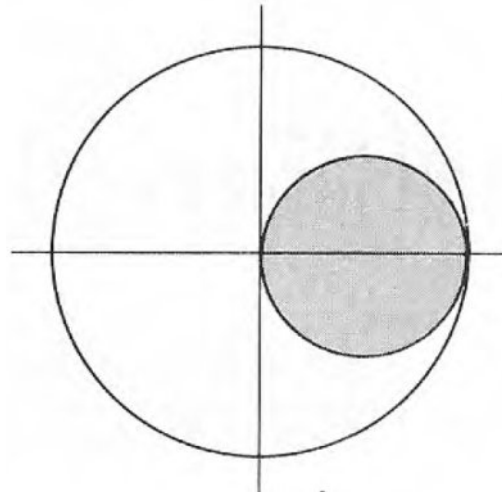
Retangular em avanço



a)

$$z = 1 + Ts$$

Retangular em atraso



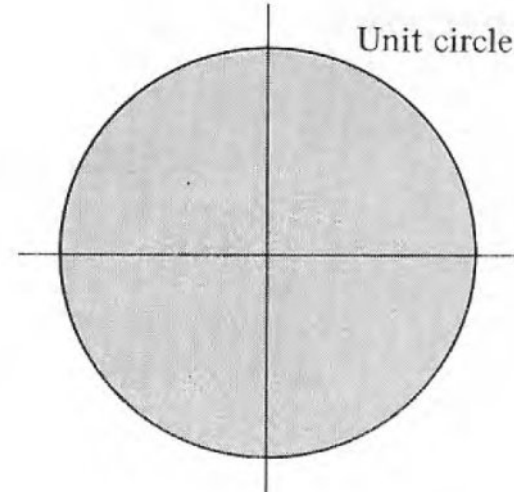
(b)

$$z = \frac{1}{2} + \left\{ \frac{1}{1 - Ts} - \frac{1}{2} \right\}$$

$$= \frac{1}{2} - \frac{1}{2} \frac{1 + Ts}{1 - Ts}$$

$$\left| z - \frac{1}{2} \right| = \frac{1}{2}$$

Trapezoidal



(c)

$$z = \frac{1 + Ts/2}{1 - Ts/2}$$

Filtro estável em s (SPE)
pode ser mapeado em um
filtro discreto instável
(pólos $|z_i| > 1$)

Equivalente Prewarp – Pré-Compensação

- Corrige a distorção que ocorre na frequência devido ao atraso $T/2$ do sistema discreto.
- Em geral, o ajuste do ganho é feito na frequência de corte do sistema → Neste ponto, ω_1 , os sistema contínuo e discreto passam a ter o mesmo ganho.

Equivalente discreto via casamento Pólo-Zero (Pole-Zero Matching)

Regras heurísticas

p/ zeros tb. $z=e^{sT}$

>> `sd=c2d(sys,T,'matched')`

1. Pólos $z=e^{sT}$
2. Zeros finitos mapeados tb. por $z=e^{sT}$
3. a) Zeros de $H(s)$ em $s = \infty$ mapeados em $H_{zp}(z)$ em -1
 $z = e^{j0} = 1 \rightarrow z = e^{j\pi} = -1$ (maior freq. possível em z)
b) se se deseja um delay:
mapear um zero em ∞ e os demais em -1
4. Ganho
 $H(s)|_{s=0} = H_{zp}(z)|_{z=1}$

Exemplo: pole-zero matching

$$H(s) = \frac{a}{s + a}$$

Solução:

Pólo em $s = -a \rightarrow$ pólo de $H(z)$ em e^{-aT}

Zero $s = \infty \rightarrow$ zero de $H(z)$ em $z = -1$

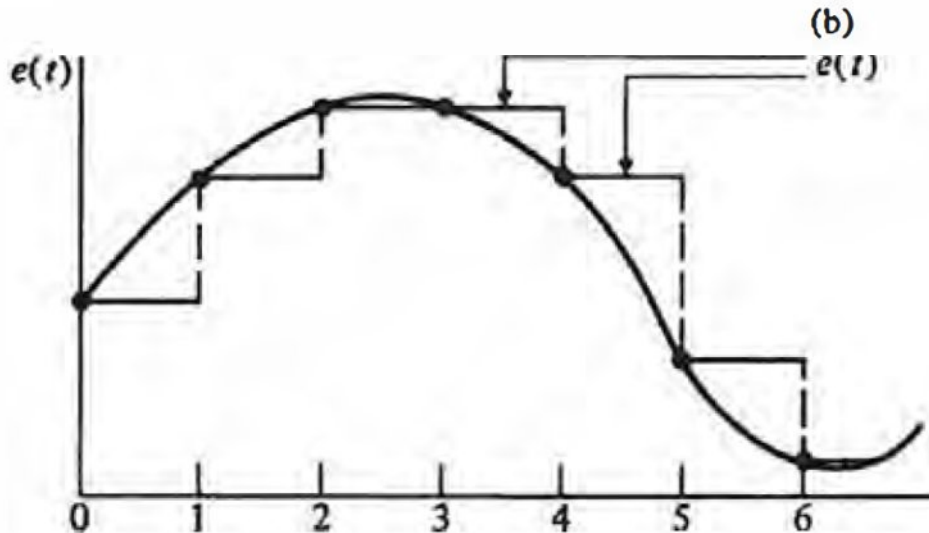
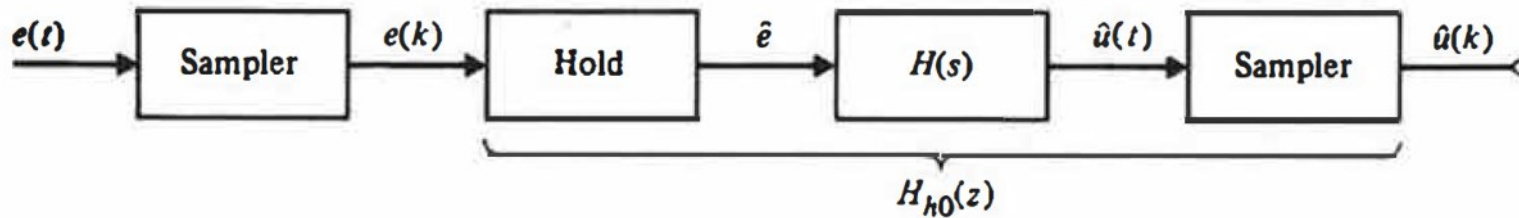
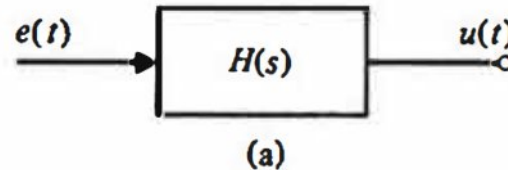
Ganho de $H(s)$ em $s = 0$ é 1 $\rightarrow$ para casar

$H(z)$ em $z = 1$ deve ter ganho $(1 - e^{-aT})/2$

$$3.a) \quad H_{zp}(z) = \frac{(z + 1)(1 - e^{-aT})}{2(z - e^{-aT})}$$

$$3.b) \quad H_{zp}(z) = \frac{1 - e^{-aT}}{z - e^{-aT}}$$

Equivalentes “Hold” (Segurador)



Equivalente segurador de ordem zero

$$H_{h0}(z) = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{H(s)}{s} \right\}$$

>> sd = c2d(sys,T,'zoh')

ou >> sd = c2d(sys,T)

Exemplo: equivalente ZOH

$$H(s) = \frac{a}{s+a}$$

$$H_{h0}(z) = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{H(s)}{s} \right\}$$

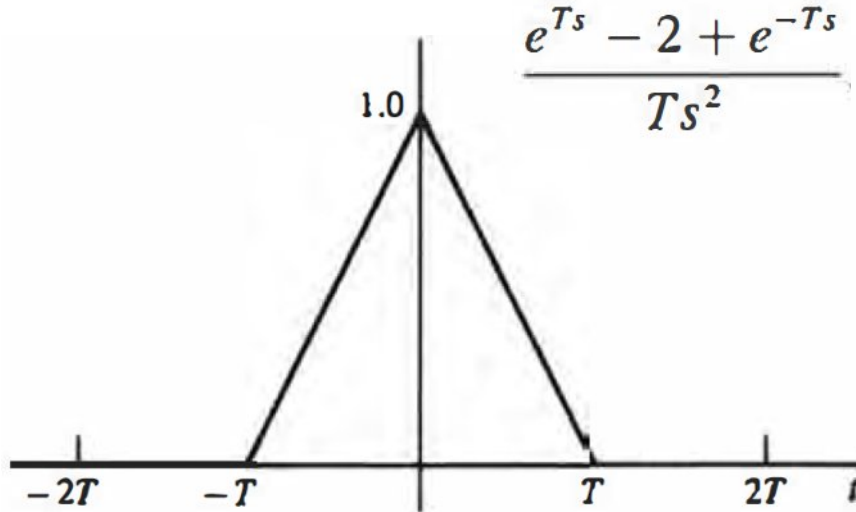
$$\frac{H(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$\mathcal{Z} \left\{ \frac{H(s)}{s} \right\} = \mathcal{Z} \left\{ \frac{1}{s} \right\} - \mathcal{Z} \left\{ \frac{1}{s+a} \right\}$$

$$\begin{aligned} \mathcal{Z} \left\{ \frac{H(s)}{s} \right\} &= \sum_0^{\infty} z^{-k} - \sum_0^{\infty} z^{-k} e^{-akT} \\ &= \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}} \\ &= \frac{(1-e^{-aT}z^{-1}) - (1-z^{-1})}{(1-z^{-1})(1-e^{-aT}z^{-1})} \end{aligned}$$


$$H_{h0}(z) = \frac{(1 - e^{-aT})}{z - e^{-aT}}$$

Equivalente “Hold” de 1ª ordem (Segurador triangular)



Resposta ao impulso:
Não-causal!!

O sistema contínuo é não causal, porém seu discreto equivalente é realizável (causal)!!

Equivalente segurador de 1ª ordem

$$H_{tri}(z) = \frac{(z-1)^2}{Tz} \mathcal{Z} \left\{ \frac{H(s)}{s^2} \right\}$$

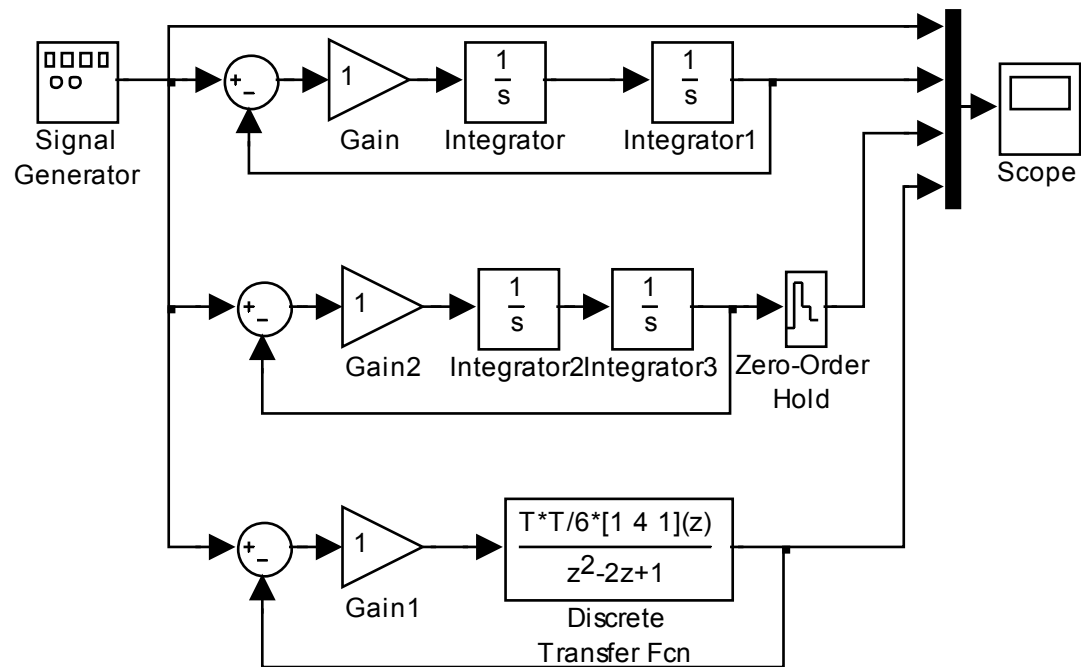
Exemplo: equivalente discreto FOH

Processo:

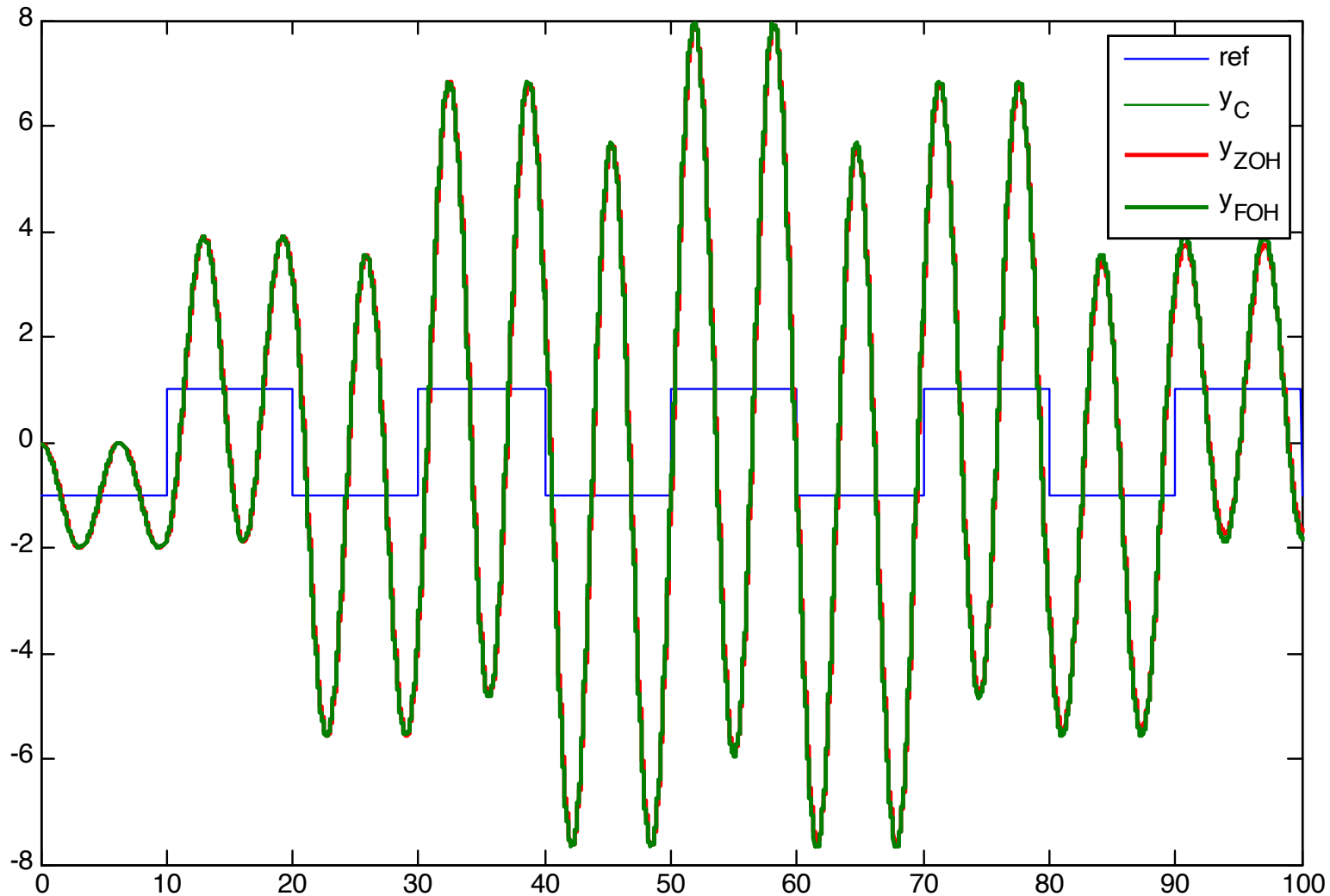
$$H(s) = 1/s^2$$

$$\mathcal{Z} \left\{ \frac{H(s)}{s^2} \right\} = \mathcal{Z} \left\{ \frac{1}{s^4} \right\} \\ = \frac{T^3}{6} \frac{(z^2 + 4z + 1)z}{(z - 1)^4}$$

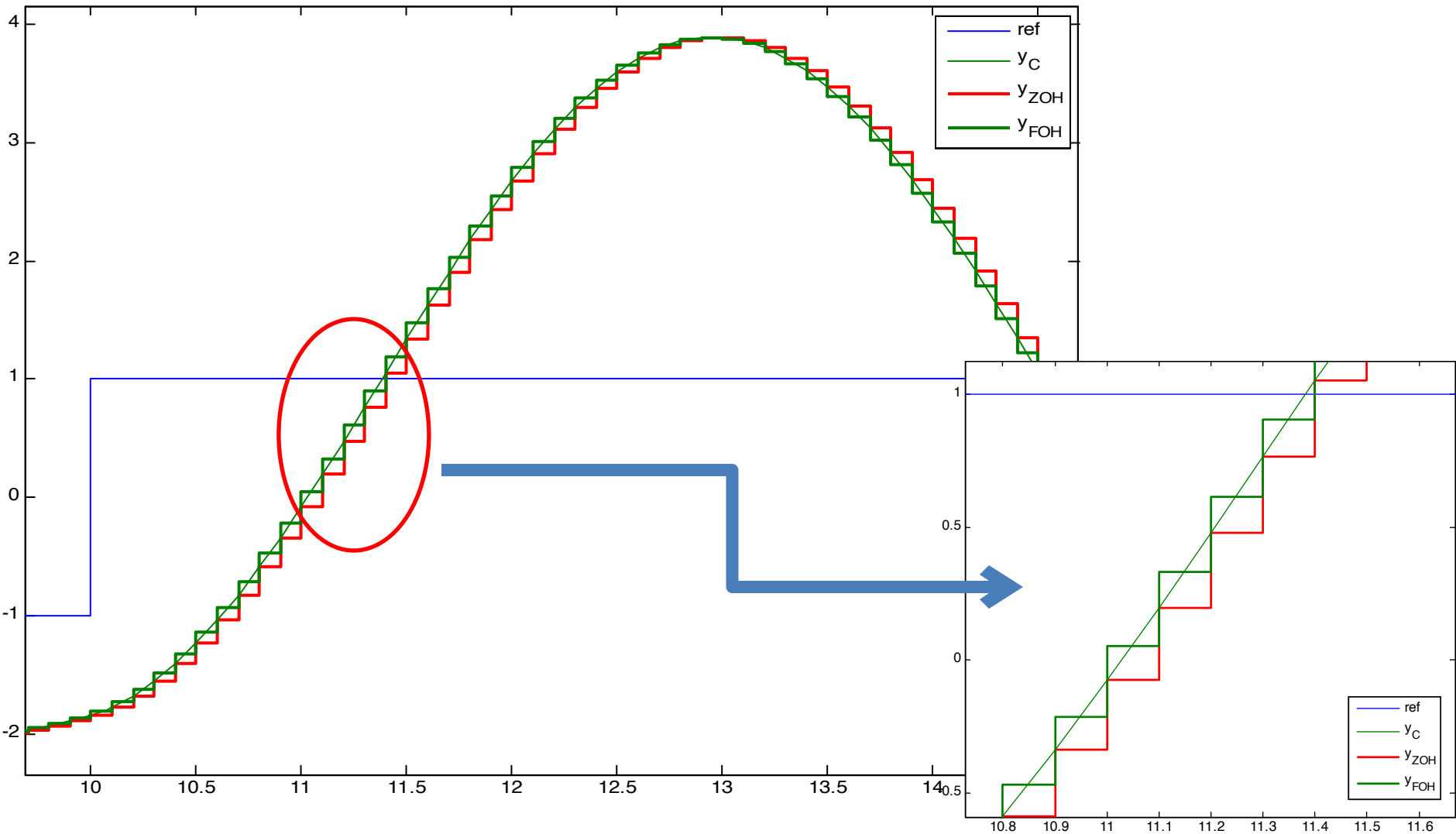
$$H_{tri}(z) = \frac{(z - 1)^2}{Tz} \frac{T^3}{6} \frac{(z^2 + 4z + 1)z}{(z - 1)^4} \\ = \frac{T^2}{6} \frac{z^2 + 4z + 1}{(z - 1)^2}$$



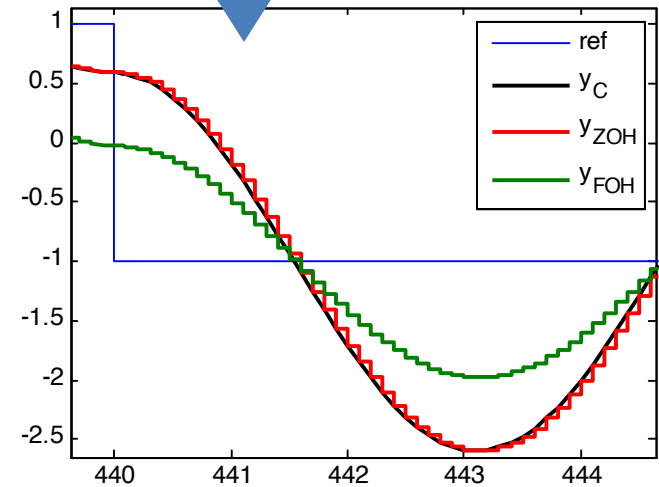
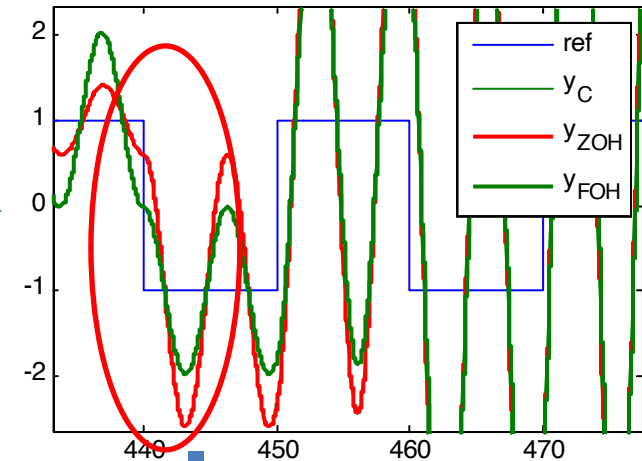
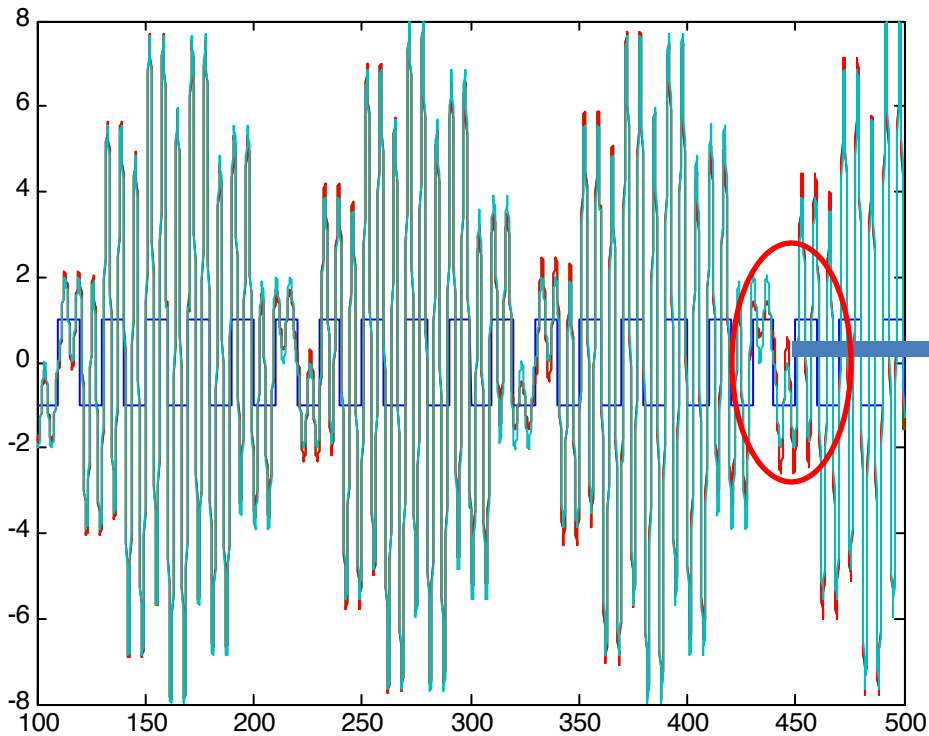
Exemplo: equivalente discreto FOH



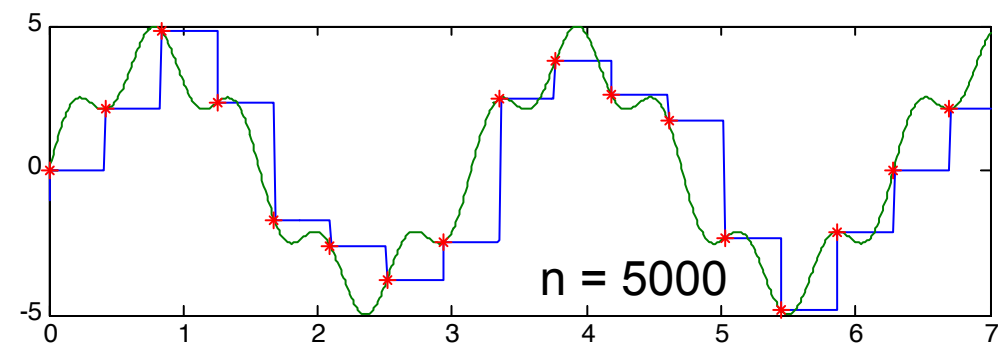
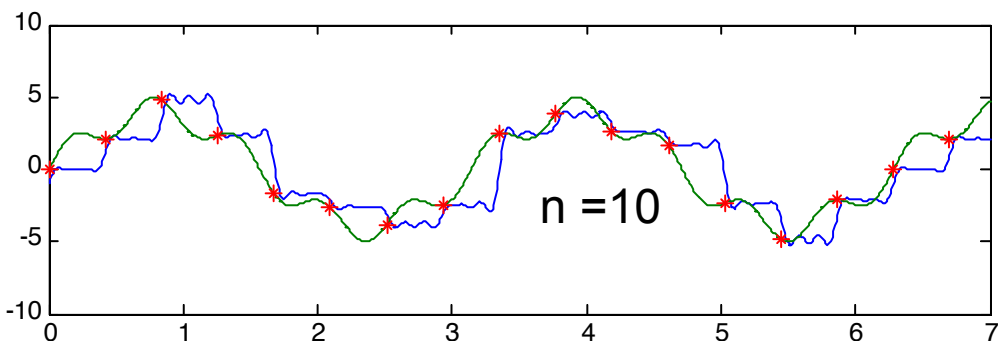
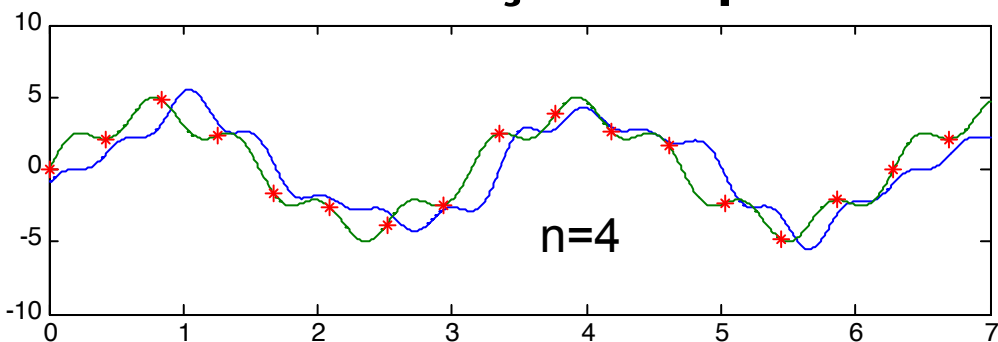
Exemplo: equivalente discreto FOH



Cuidado com a extrapolação !!



Reconstrução a partir das amostras



```
>> reconstr(n);
```

$$y = A1 \cdot \sin(w1 \cdot t + a1) + A2 \cdot \sin(w2 \cdot t + a2)$$

```
A1=4; A2=1;
w1=2; w2=10;
a1=pi/5; a2=pi/16;
a1=0; a2=0;
ws=15;
```

```
for i=1:4:n-1
```

```
  y =
```

```
    A1*cos(a1)*sinc((w(i)*T/2)/pi)*sin(w(i)*t-w(i)*T/2)...
+ A1*sin(a1)*sinc((w(i)*T/2)/pi)*cos(w(i)*t-w(i)*T/2)...
+ A1*cos(a1)*sinc((w(i+2)*T/2)/pi)*sin(w(i+2)*t-w(i+2)*T/2-pi)...
+ A1*sin(a1)*sinc((w(i+2)*T/2)/pi)*cos(w(i+2)*t-w(i+2)*T/2)...
+ A2*cos(a2)*sinc((w(i+1)*T/2)/pi)*sin(w(i+1)*t-w(i+1)*T/2)...
+ A2*sin(a2)*sinc((w(i+1)*T/2)/pi)*cos(w(i+1)*t-w(i+1)*T/2)...
+ A2*cos(a2)*sinc((w(i+3)*T/2)/pi)*sin(w(i+3)*t-w(i+3)*T/2-pi)...
+ A2*sin(a2)*sinc((w(i+3)*T/2)/pi)*cos(w(i+3)*t-w(i+3)*T/2);
```

```
...
end
```

Especificação de projeto em Z

(Dinâmica dominante de 2ª ordem sem zeros)

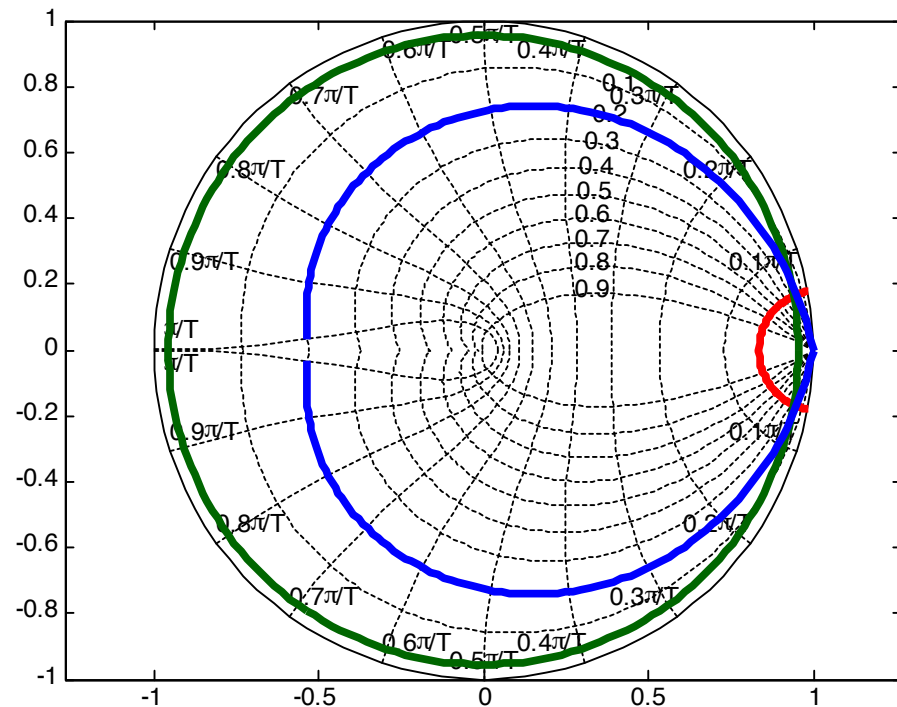
`spec_z(zeta, tr, ts, T)` - Mostra no plano z a região que atende a especificação

zeta - fator de amortecimento ($\%UP = \exp(-\pi \cdot \text{zeta}) / \sqrt{1 - \text{zeta}^2}$)

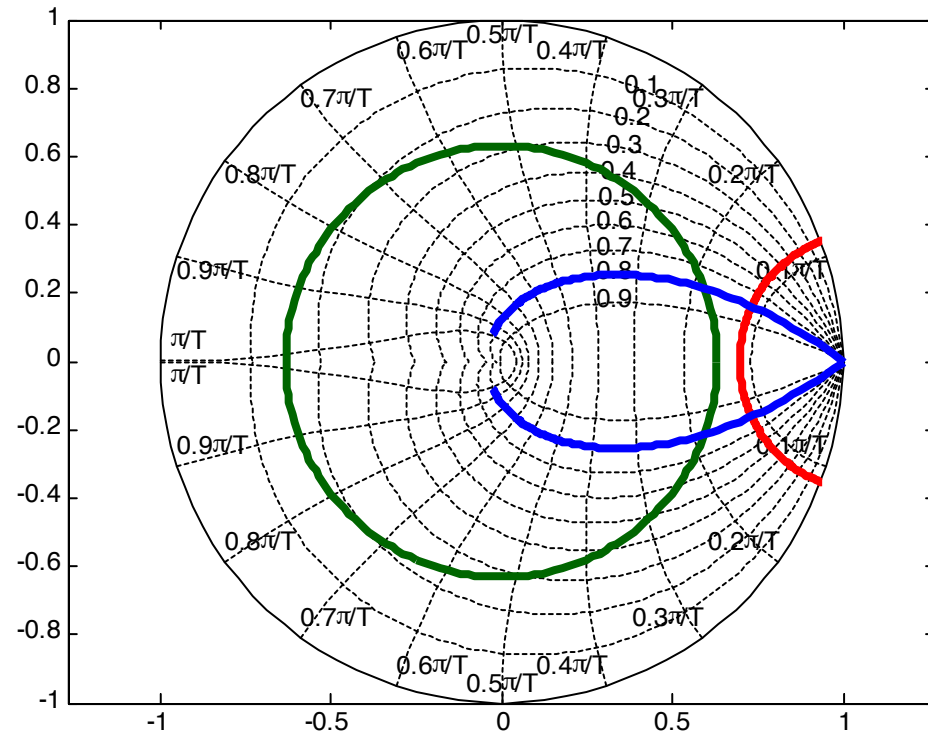
tr - tempo de subida 10 - 90% ($\omega_n \geq 1.8 / tr$)

ts - tempo de acomodação 1% ($\sigma \geq 4.6 / ts$)

T - taxa de amostragem

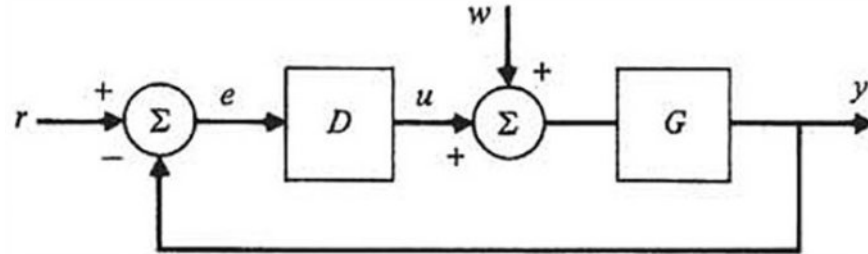


`spec_z(0.2, 1, 10, 1)`

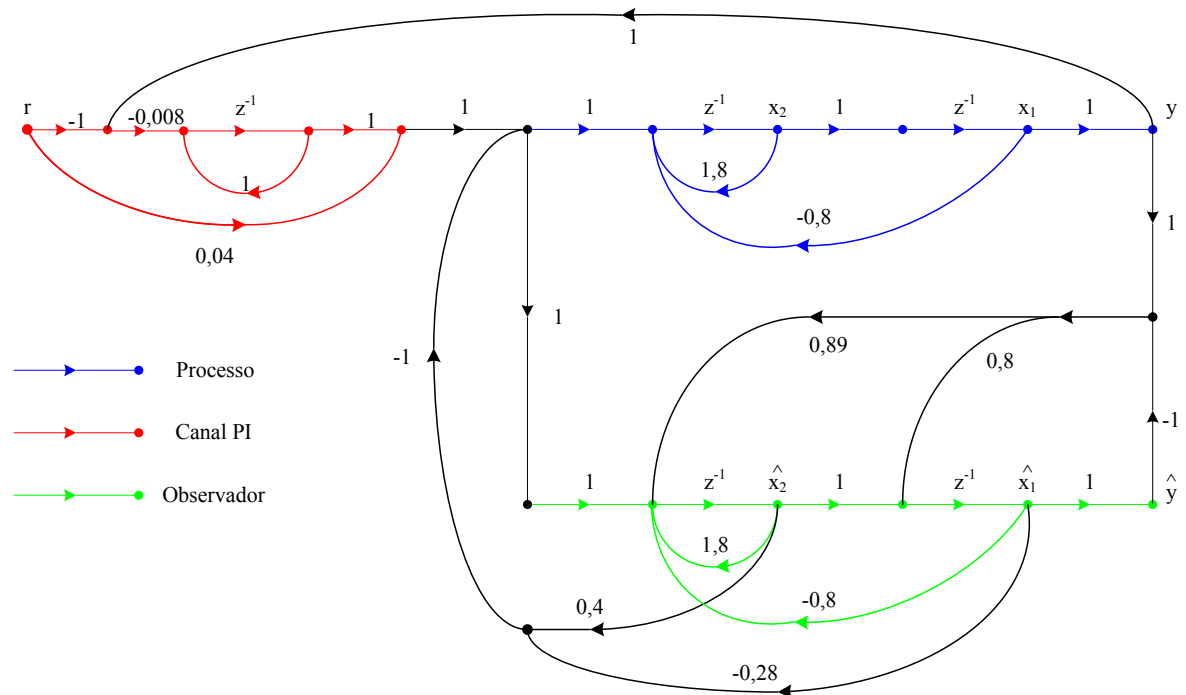


7. Projeto utilizando Transformadas

- Controle Clássico
 - Transformada de Laplace
 - Transformada Z



- Controle Moderno
 - Espaço de Estados



7. Projeto utilizando Transformadas

Especificações (aprox. de 2ª ordem, sem zeros)

- Tempo de subida $\leftrightarrow$ frequência natural

$$\omega_n \geq \frac{1.8}{t_r}$$

- Tempo de acomodação (1%)

$$|Re\{s_i\}| = \sigma = \zeta\omega_n \geq 4.6/t_s$$

- Sobressinal $\leftrightarrow$ Fator de amortecimento

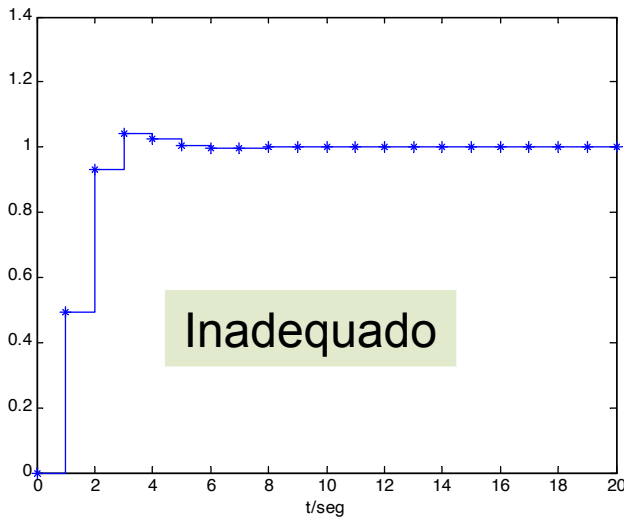
$$M_p = 100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

- Coeficiente de Erro de Velocidade

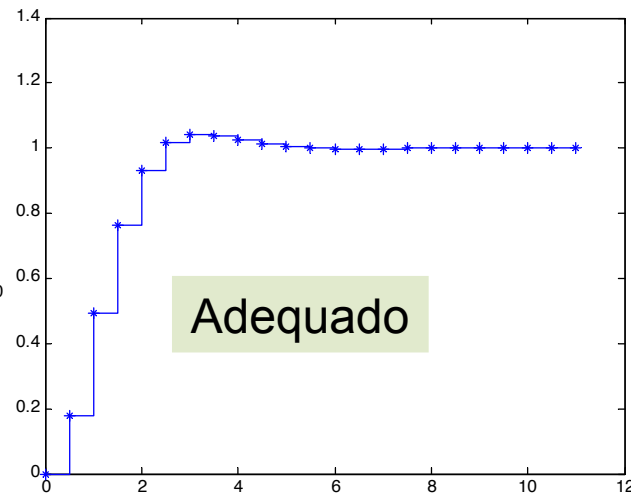
$$K_v = \frac{r_0}{e_{ss}} = \lim_{s \rightarrow 0} sD(s)G(s)$$

Escolha da Taxa de Amostragem

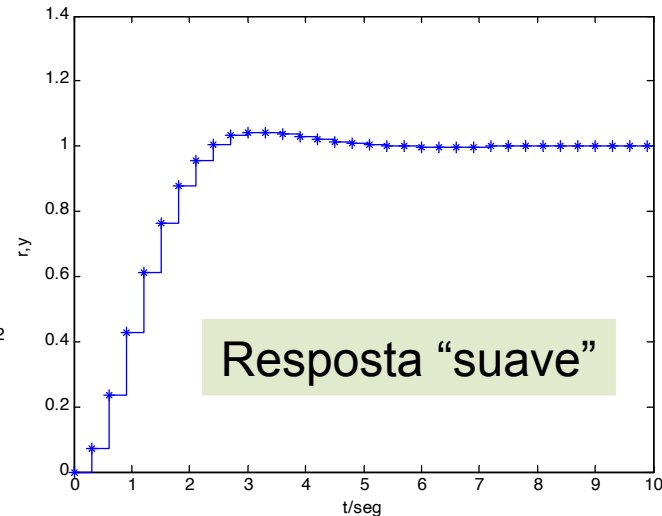
T deve levar, em Malha Fechada, a 6 amostras entre 0 e t_r



$T=1\text{s}$; 3 amostras (0 e t_r)



$T=0,5\text{ s}$; 6 amostras (0 e t_r)



$T=0,3\text{ s}$; 10 amostras (0 e t_r)

$$\frac{2}{s^2 + 2s + 2}$$

Exemplo: Escolha da Taxa de Amostragem

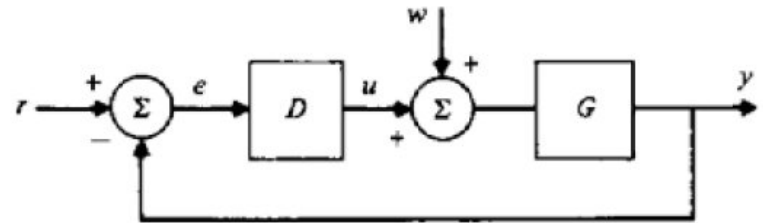
Relação entre a frequência de amostragem, ω_s , e a frequência natural, ω_n , para que $N=10$ (0 e t_r) ?

$$\omega_s = 2\pi/T$$

$$\omega_n = 1.8/t_r$$

$$= \frac{1.8}{10T}$$

$$\omega_n = \frac{0.18\omega_s}{2\pi} \quad \omega_n \approx \frac{\omega_s}{35}$$



(aprox. de 2ª ordem, sem zeros)

$$\omega_n = \frac{1.8}{8T} \quad \omega_n = \frac{0.225\omega_s}{2\pi} \quad \omega_n \approx \frac{\omega_s}{28}$$

A taxa de amostragem deve ser 28 a 35 vezes a frequência natural !

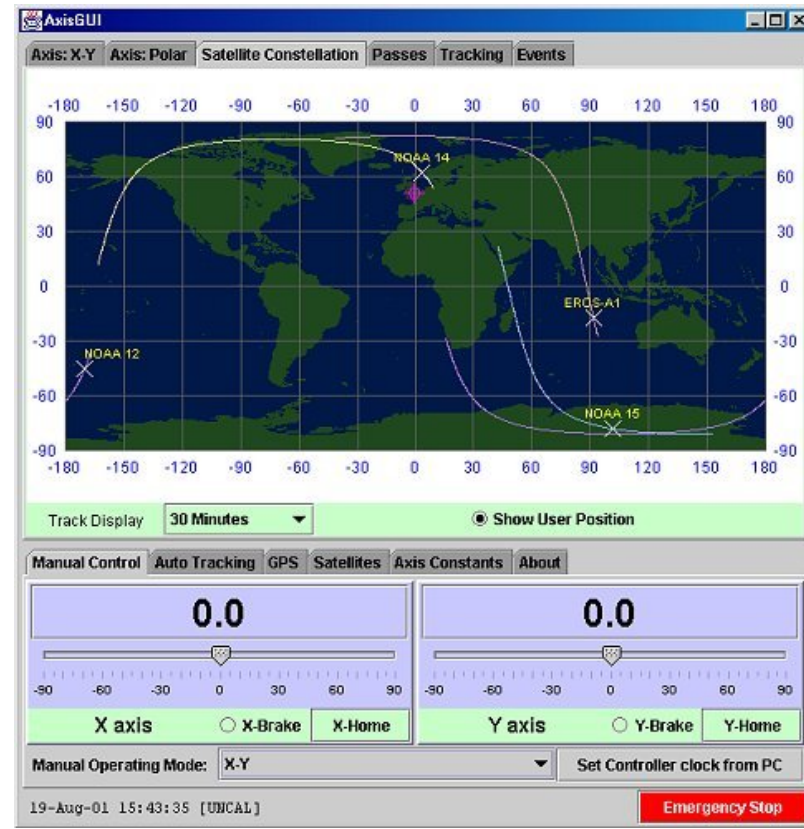
Exemplo – Projeto Antena – rastreador angular

Especificações

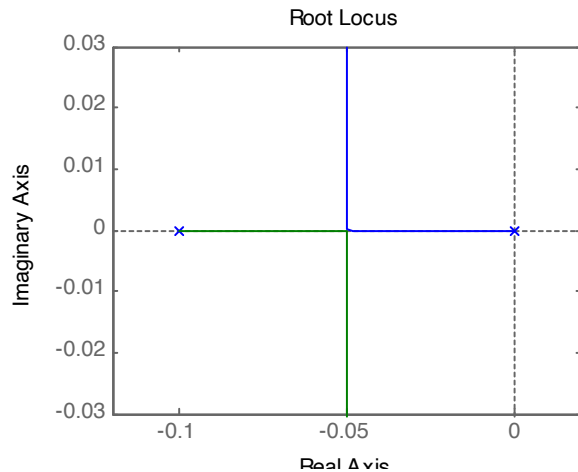
- Sobrepasso, degrau de entrada: 16%
- Tempo de acomodação (1%) < 10 seg
- e_{ss} rastreamento rampa de inclinação 0,01 rad/seg menor que 0,01 rad
- 10 amostras dentro de t_r



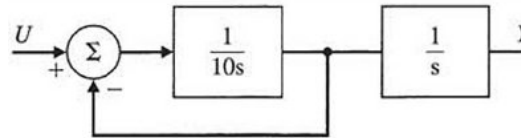
$$G(s) = \frac{1}{s(10s + 1)}$$



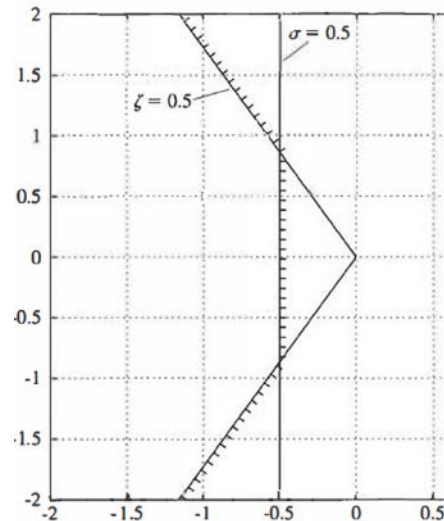
Projeto Controle Antena



Controle P



$$G(s) = \frac{1}{s(10s + 1)}$$



Projeto contínuo (em s):

Compensador em Avanço
Cancelar polo em 0,1

$$D(s) = \frac{10s + 1}{s + 1}$$

Solução

$$M_p \leq 16\% \rightarrow \zeta = 0.5$$

$$\sigma \geq \frac{4.6}{10} \rightarrow \sigma \geq 0.46$$

$$e_{ss} \leq \frac{0.01}{0.01} = 1 \rightarrow K_v = \frac{1}{e_{ss}} \geq 1$$

Projeto Controle Antena

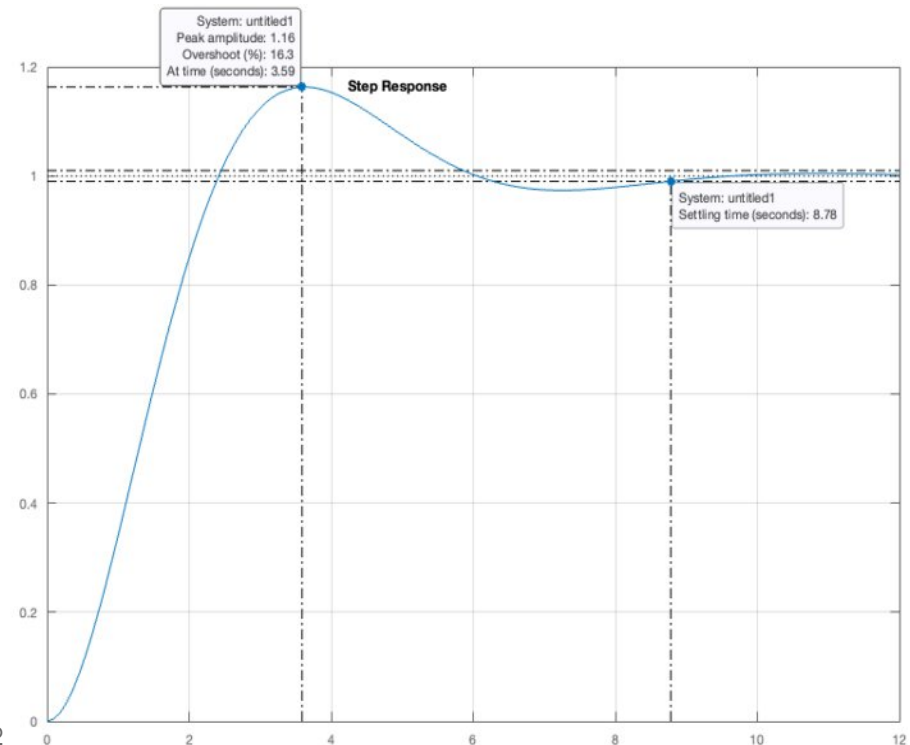
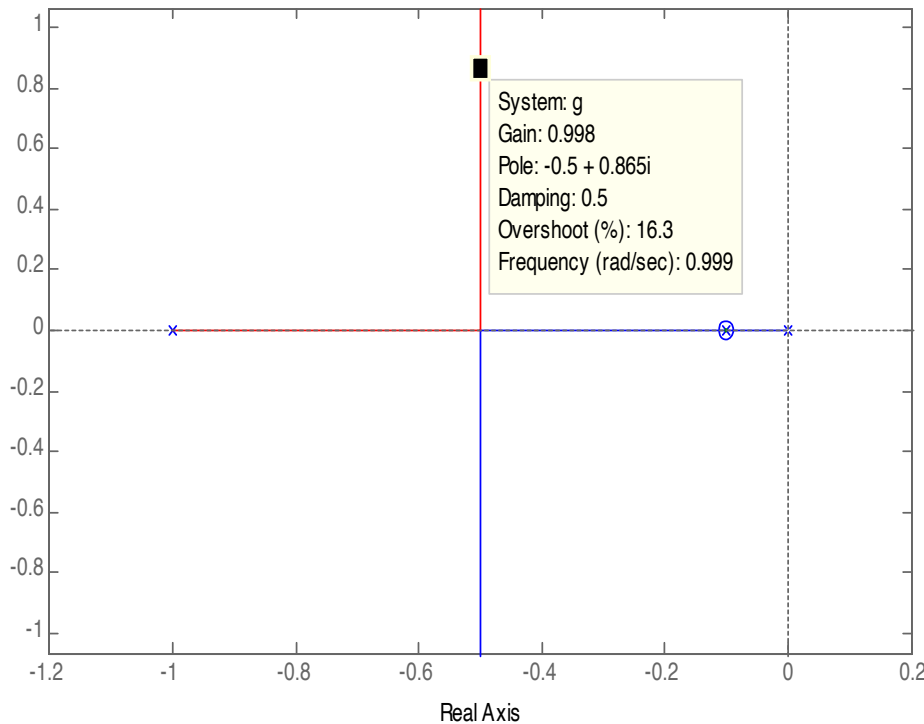
$$G(s) = \frac{1}{s(10s + 1)}$$

Projeto contínuo (em s):

Compensador em Avanço
Cancelar polo em 0,1

$$D(s) = \frac{10s + 1}{s + 1}$$

Root Locus



Controle Digital Antena - Emulação

Freq. natural $\omega_n = 1$

$t_r = 1,8 \text{ seg}$

$T = t_r / 10 = 0,18 \text{ seg} \rightarrow T = 0,2 \text{ seg}.$

$$G(s) = \frac{1}{s(10s + 1)} \quad D(s) = \frac{10s + 1}{s + 1}$$

Implementação discreta por emulação: *Pole-Zero Matcing*

$$D(z) = K \frac{z - z_1}{z - p_1}$$

$$z_1 = e^{-0,1,0,2} = 0,9802$$

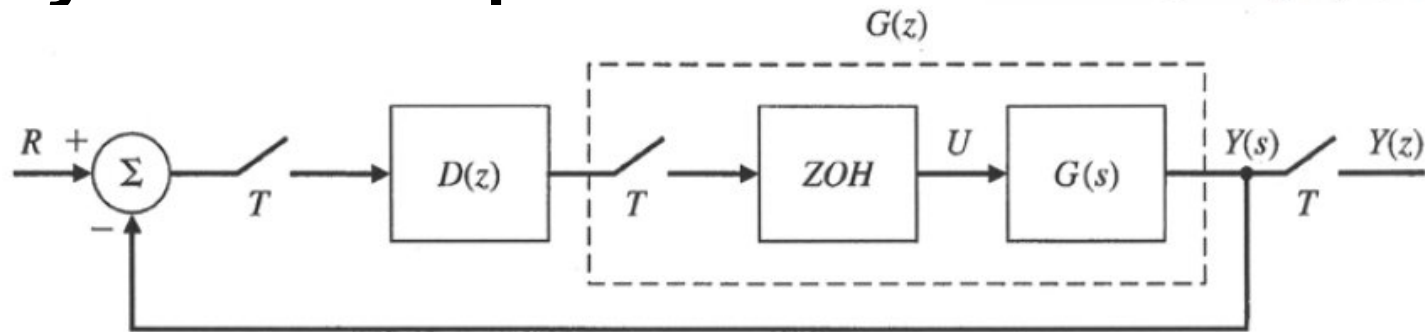
$$p_1 = e^{-1,0,2} = 0,8187$$

Ganho DC: $\lim_{z \rightarrow 1} D(z) = \lim_{s \rightarrow 0} D(s) = 1 = K \frac{1 - 0,9802}{1 - 0,8187} \Rightarrow K = 9,15$

$$D(z) = 9,15 \frac{z - 0,9802}{z - 0,8187}$$

Projeto Completo

$$G(z) = \frac{z-1}{z} \mathcal{Z} \left\{ \frac{a}{s^2(s+a)} \right\}$$



$$G(z) = \frac{z-1}{z} \mathcal{Z} \left\{ \frac{1}{s^2} - \frac{1}{as} + \frac{1}{a} \frac{1}{s+a} \right\}$$

$$G(z) = \frac{z-1}{z} \left\{ \frac{Tz}{(z-1)^2} - \frac{z}{a(z-1)} + \frac{1}{a} \frac{z}{z-e^{-aT}} \right\}$$

$$= \frac{Az+B}{a(z-1)(z-e^{-aT})},$$

$$A = e^{-aT} + aT - 1, \quad B = 1 - e^{-aT} - aTe^{-aT}.$$

$$T = 0.2 \text{ and } a = 0.1 \quad \Rightarrow \quad G(z) = 0.00199 \frac{z + 0.9934}{(z-1)(z-0.9802)}$$

Implementação do Controlador Digital

$$D(z) = 9,15 \frac{z - 0,9802}{z - 0,8187}$$

$$D(z) = \frac{U(z)}{E(z)} = 9.15 \frac{1 - 0.9802z^{-1}}{1 - 0.8187z^{-1}}$$

$$(1 - 0.8186z^{-1})U(z) = 9.15(1 - 0.9802z^{-1})E(z)$$

Equação de recorrência:

$$u(k) = 0.8186u(k-1) + 9.15(e(k) - 0.9802e(k-1))$$

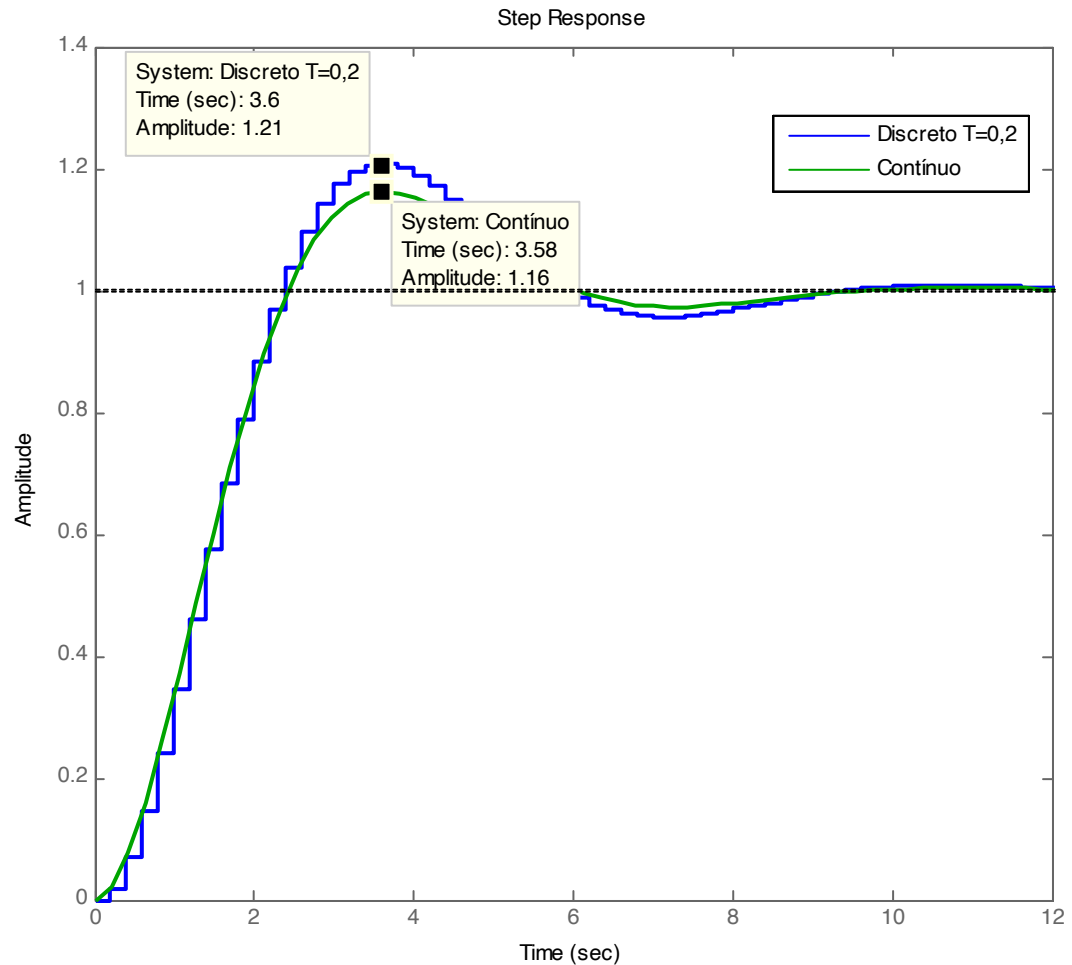
Simulação do Projeto por emulação

T=0,2 seg

$$D(s) = \frac{10s + 1}{s + 1}$$

$$G(z) = 0.00199 \frac{(z + 0.9934)}{(z - 1)(z - 0.9802)}$$

$$D(z) = 9.15 \frac{(z - 0.9802)}{(z - 0.8187)}$$



21% de sobresinal em vez dos 16% especificados !

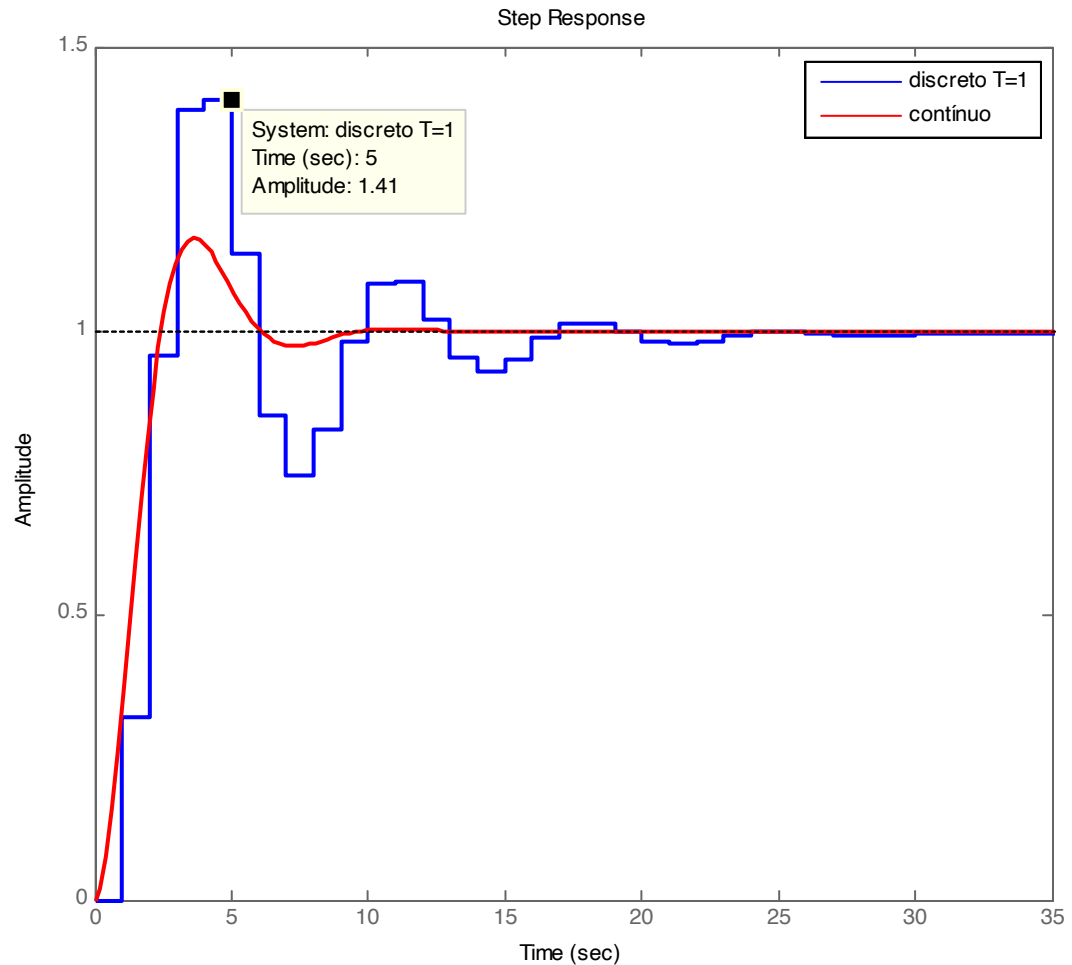
Simulação do Projeto por emulação

T=1 seg

$$D(s) = \frac{10s + 1}{s + 1}$$

$$G(z) = 0.048374 \frac{(z + 0.9672)}{(z - 1)(z - 0.9048)}$$

$$D(z) = 6.64 \frac{(z - 0.9368)}{(z - 0.3679)}$$



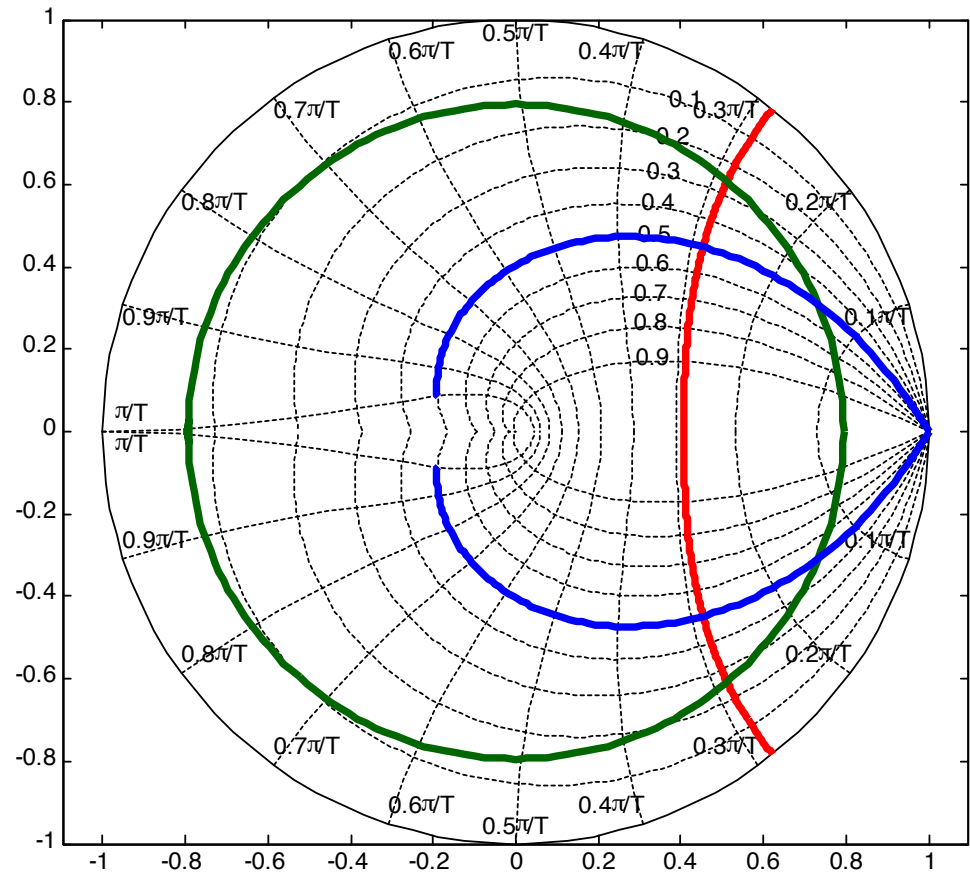
41% de sobresinal em vez dos 16% especificados !

Projeto LGR direto no plano Z

ω_n, ζ, M_p (sist. contínuo)

$$\sigma = \zeta\omega_n \Rightarrow r = e^{-\sigma T}$$

zgrid



Projeto digital Rastreamento Antena

$$M_p \leq 16\% \rightarrow \zeta = 0.5$$

$$\sigma \geq \frac{4.6}{10} \rightarrow \sigma \geq 0.46$$

$$e_{ss} \leq \frac{0.01}{0.01} = 1 \rightarrow K_v = \frac{1}{e_{ss}} \geq 1$$

$$\omega_n = N\pi / 10T$$

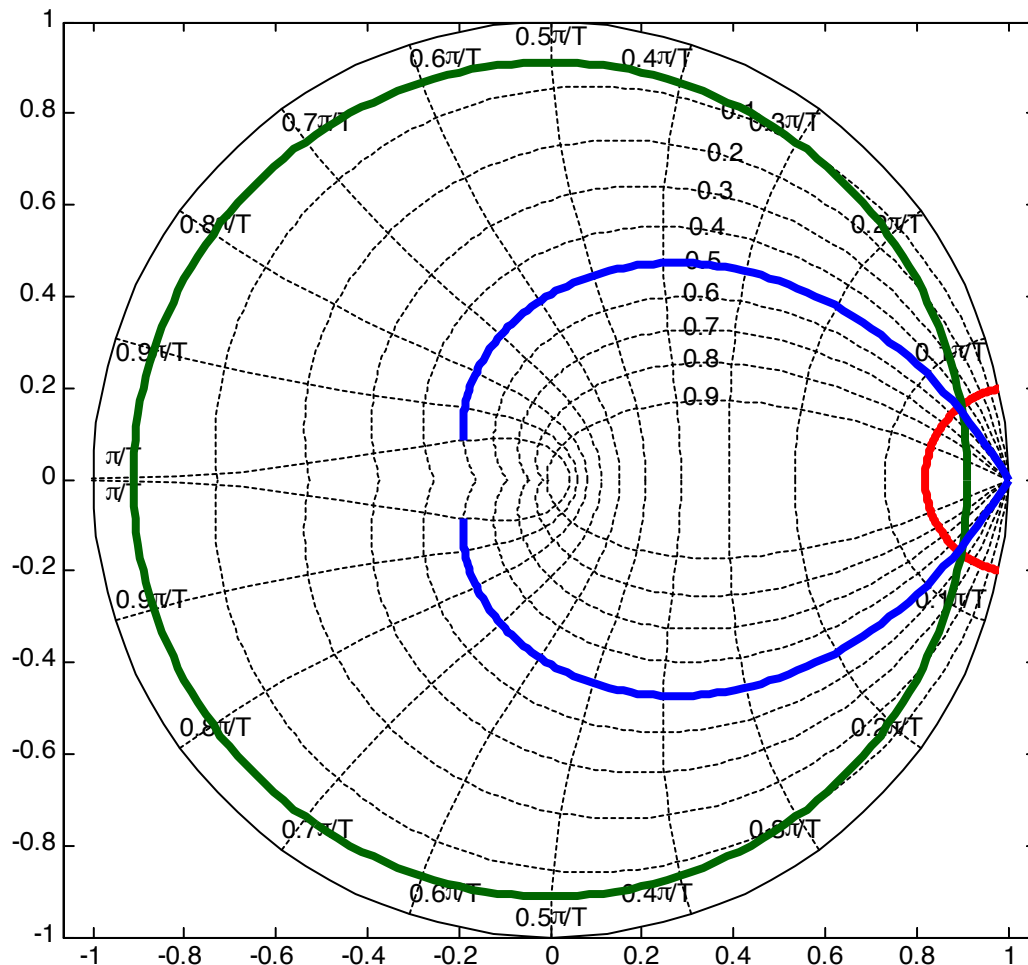
$$N = 10T\omega_n / \pi = 2 / \pi = 0,64$$

$$r = e^{-0,5T} = 0,9048$$

»zgrid():

N=1:10, 10 linhas de ω_n ctes

0,... pi/T



Coeficientes de Erro

tipo 0

$$G(z) = (1 - z^{-1})Z \left\{ \frac{G(s)}{s} \right\}$$

$$E(z) = \frac{R(z)}{1 + D(z)G(z)} = \frac{z}{z-1} \frac{1}{1 + D(z)G(z)}$$

$$\begin{aligned} e(t \rightarrow \infty) &= \lim_{z \rightarrow 1} (z-1) \frac{z}{z-1} \frac{1}{1 + D(z)G(z)} \\ &= \frac{1}{1 + D(1)G(1)} \end{aligned}$$

$$e(t \rightarrow \infty) = \frac{1}{1 + K_p}$$

tipo 1 (DG tem polo em $z = 1$)

$$E(z) = \frac{Tz}{(z-1)^2} \frac{1}{1 + D(z)G(z)}$$

$$\begin{aligned} e(t \rightarrow \infty) &= \lim_{z \rightarrow 1} (z-1) \frac{Tz}{(z-1)^2} \frac{1}{1 + D(z)G(z)} \\ &= \lim_{z \rightarrow 1} \frac{Tz}{(z-1)(1 + D(z)G(z))} \stackrel{\Delta}{=} \frac{1}{K_v} \end{aligned}$$

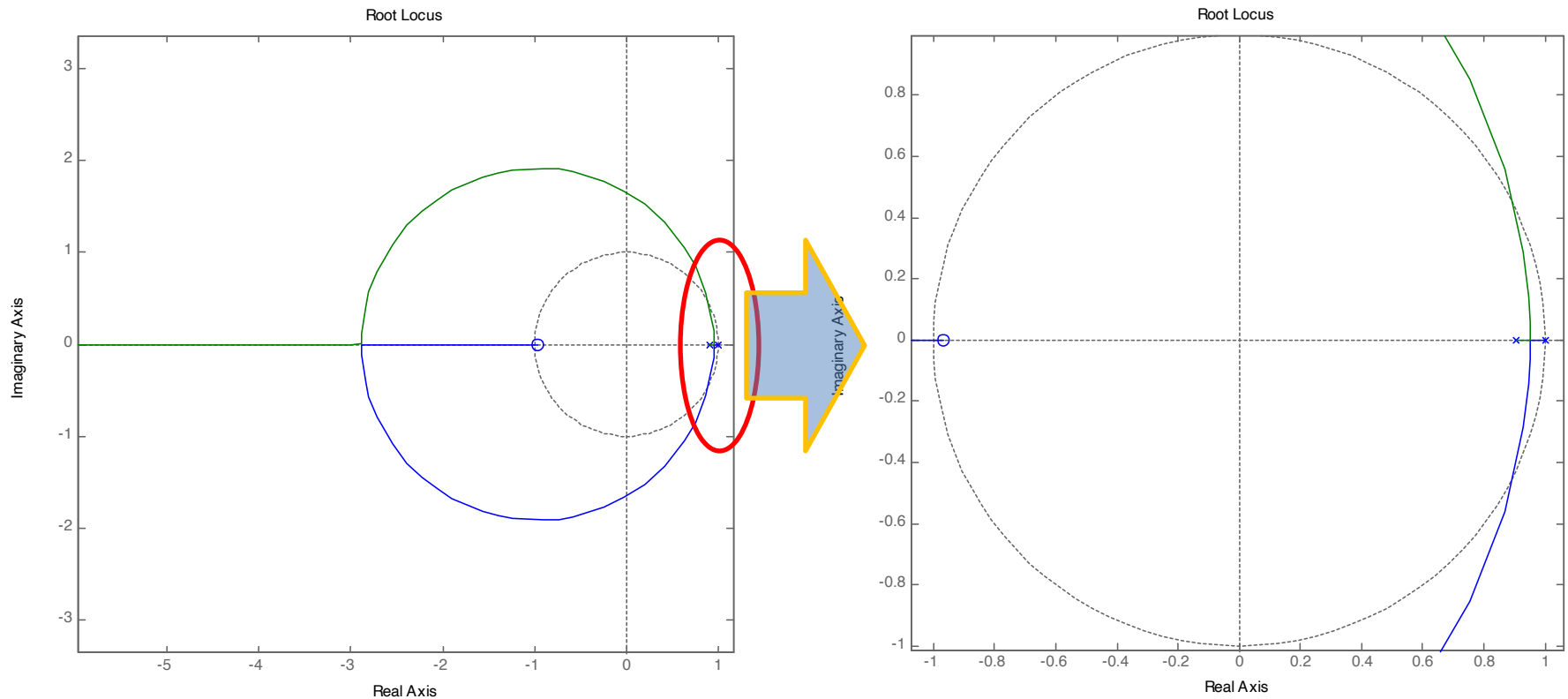
$$K_v = \lim_{z \rightarrow 1} \frac{(z-1)D(z)G(z)}{Tz}$$

LGR Discreto

$$1 + D(z)G(z) = 0$$

$$G(z) = 0.048374K \frac{(z + 0.9672)}{(z - 1)(z - 0.9048)}$$

Controle Proporcional: muito lento e/ou oscilatório!



Controle Proporcional

- O raio dos pólos é sempre maior que 0,95

$$r = e^{-0,5T} = 0,9048 \quad \rightarrow \text{necessário compensador dinâmico}$$

$$K = 19 \rightarrow K_v = 0,92 \quad \rightarrow \text{instável}$$

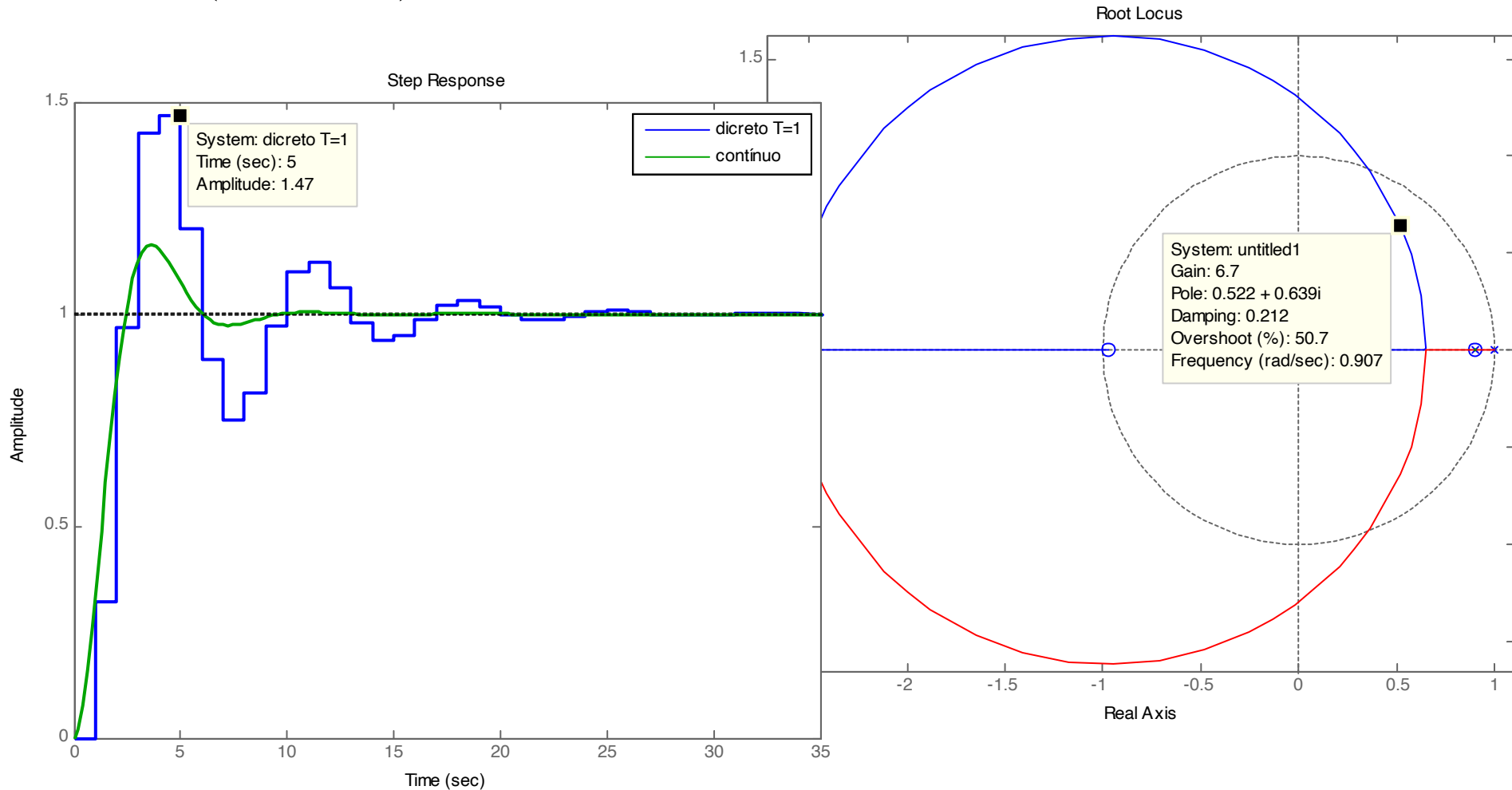
$$K_v = \lim_{z \rightarrow 1} \frac{(z-1)D(z)G(z)}{Tz} = \lim_{z \rightarrow 1} \frac{0.0484(z+0.9672)}{Tz(z-0.9048)}$$

$$G(z) = 0.048374K \frac{(z+0.9672)}{(z-1)(z-0.9048)}$$

$$G(z) = 0.048374K \frac{(z + 0.9672)}{(z - 1)(z - 0.9048)}$$

Compensador em Avanço

$$D(z) = \frac{(z - 0.9048)}{(z - 0.3679)} \rightarrow \text{Emulação do Compensador em Avanço, } T=1 \text{ seg, } K=6,64$$



Especificações no plano Z

$$\sigma \geq \frac{4.6}{10} \rightarrow \sigma \geq$$

$$e_{ss} \leq \frac{0.01}{0.01} = 1 \rightarrow K_v =$$

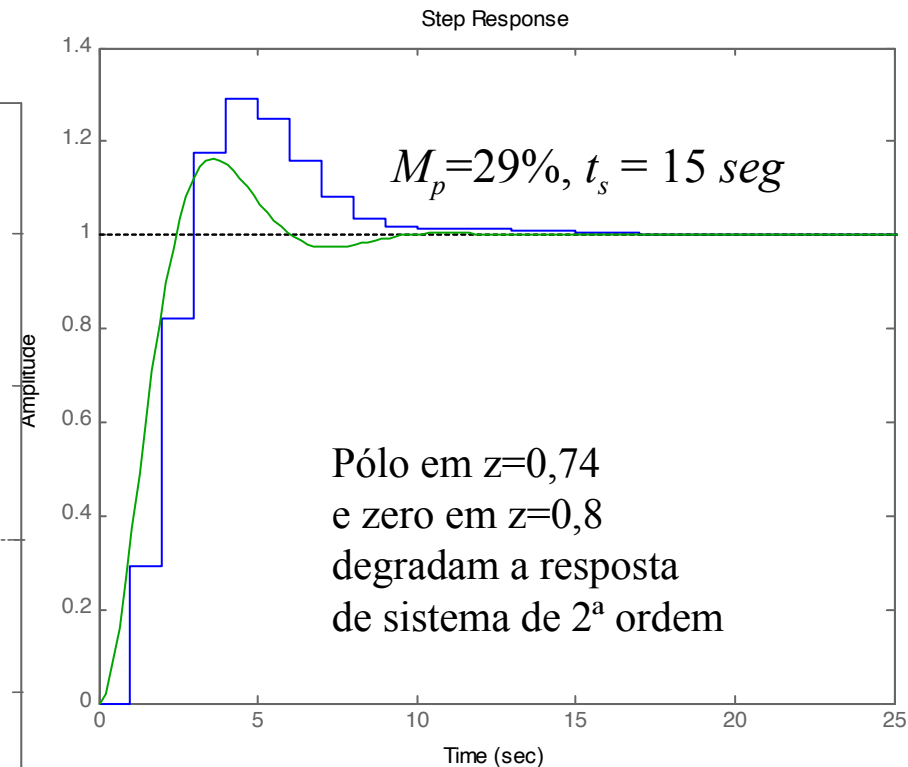
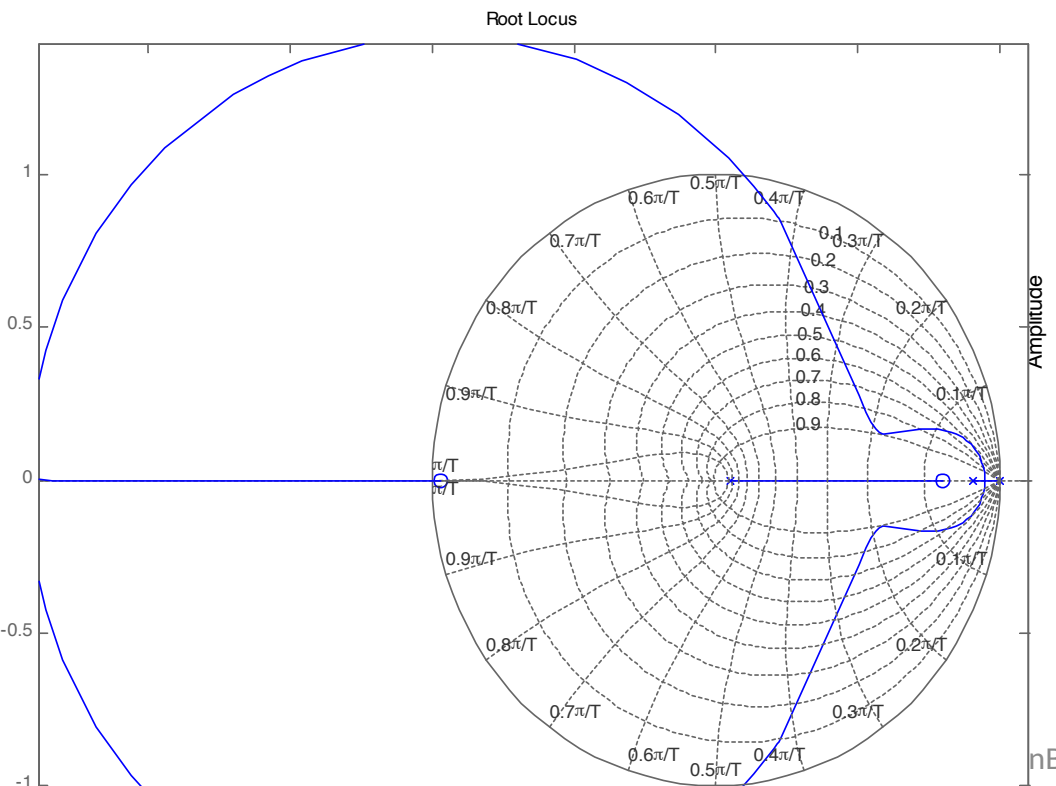
- Trazer os pólos mais para a esquerda → aumentar o avanço.

$$K_v \rightarrow 1 \rightarrow K_v = \lim_{z \rightarrow 1} \frac{(z-1)D(z)G(z)}{Tz}$$

$$t_s \leq 10s \rightarrow r^{-0.5.1} = 0,61$$

$$\xi = 0,5 \rightarrow \text{espiral de } \xi = 0,5$$

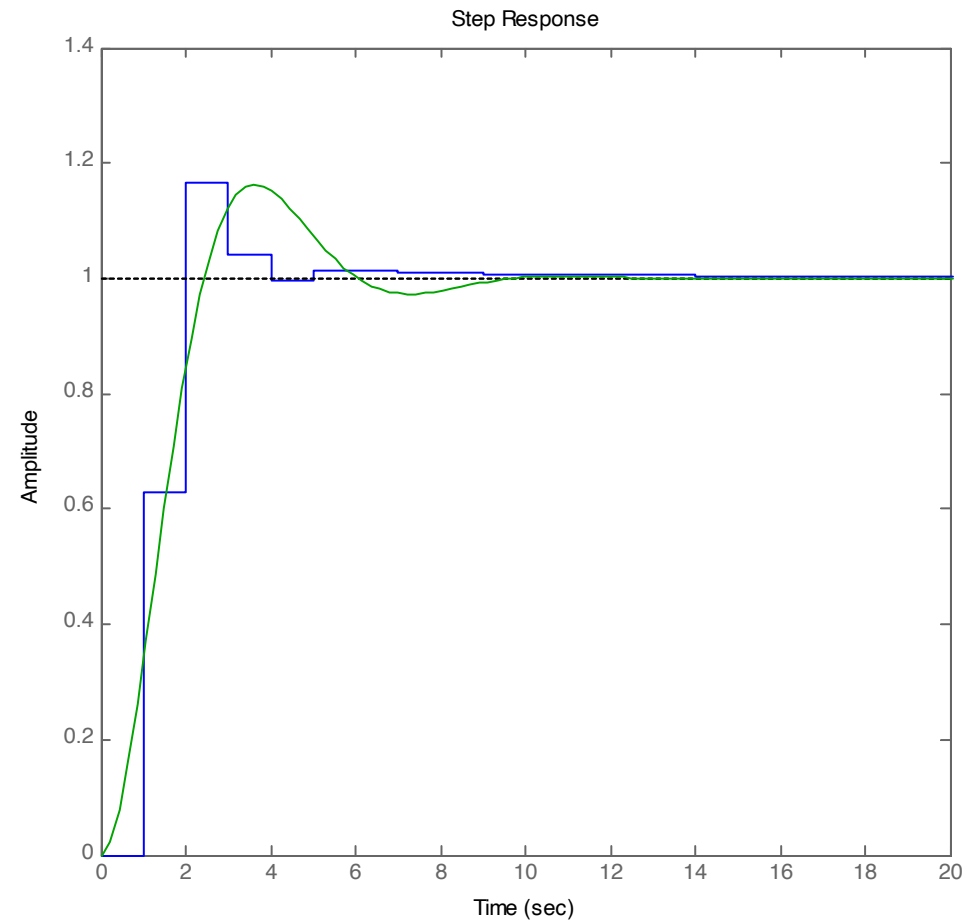
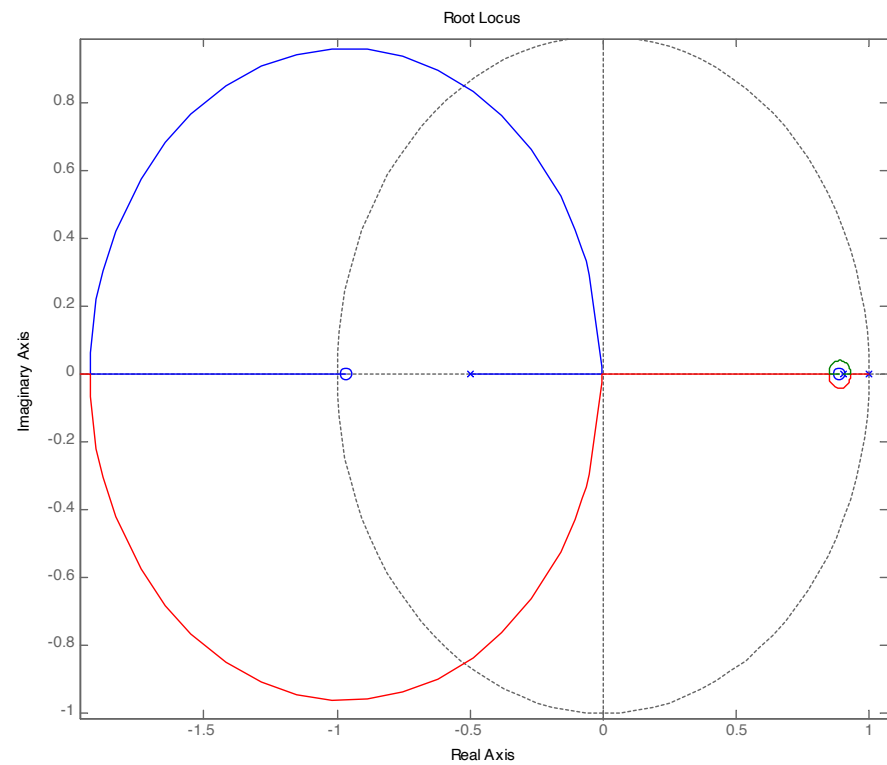
$$D(z) = 6 \frac{(z-0.8)}{(z-0.05)} \quad K_v = 1,26$$



Ajuste Fino

$$D(z) = 13 \frac{(z - 0.88)}{(z + 0.5)}$$

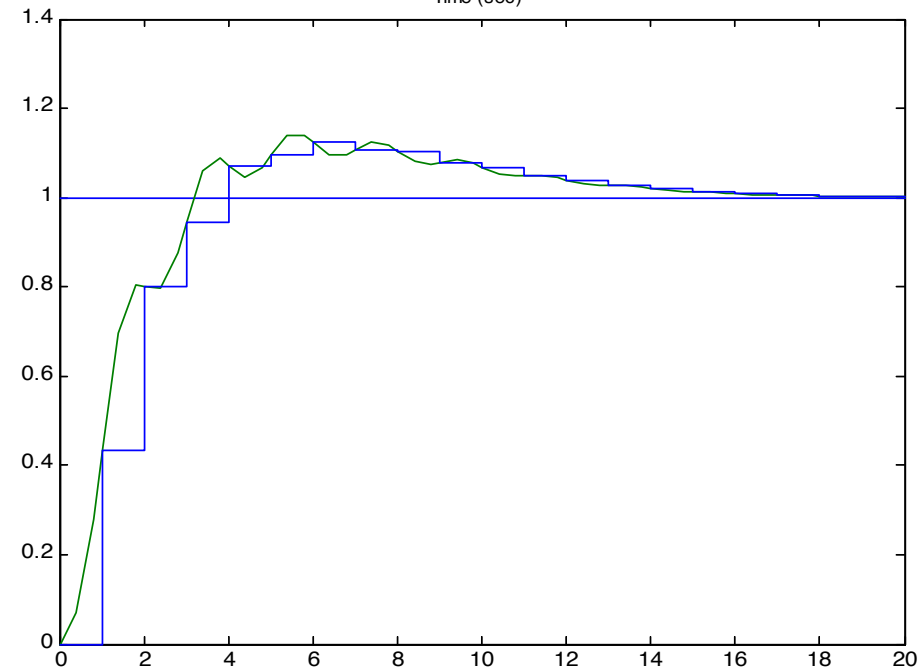
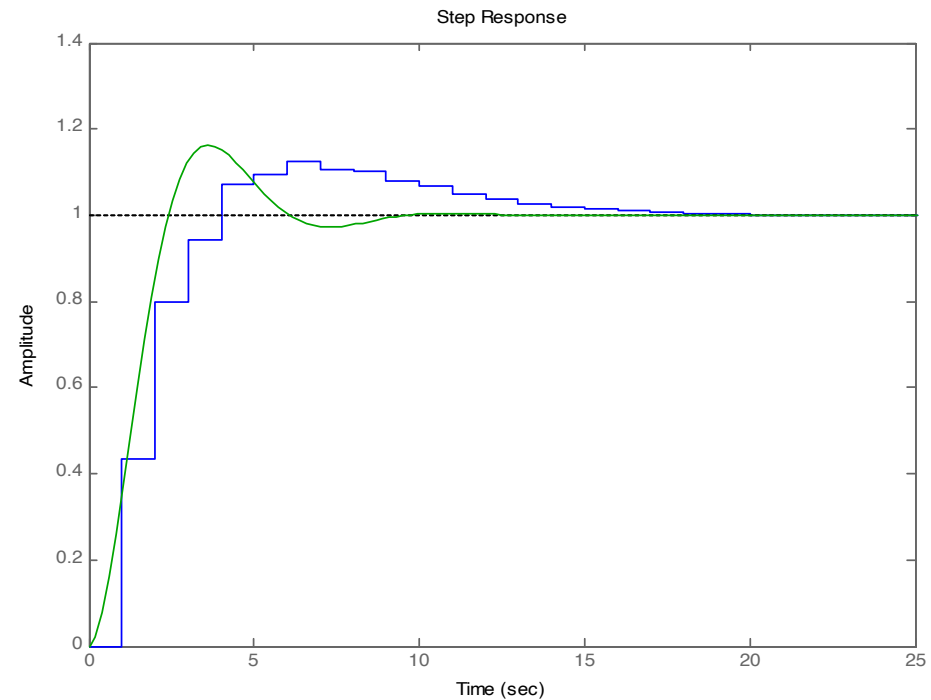
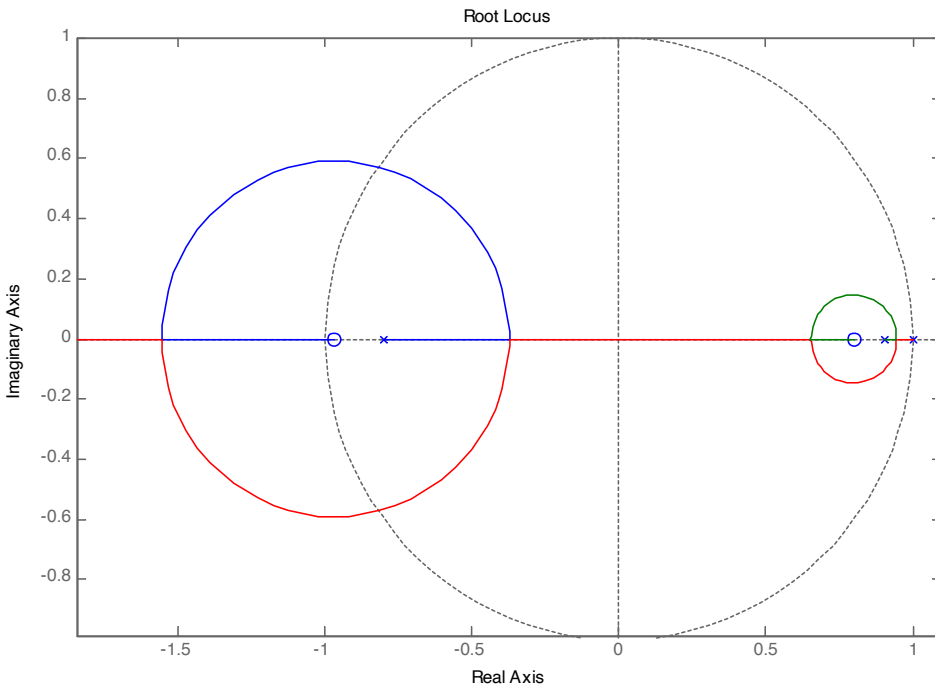
$$K_v = 1,04$$



Cuidado!!

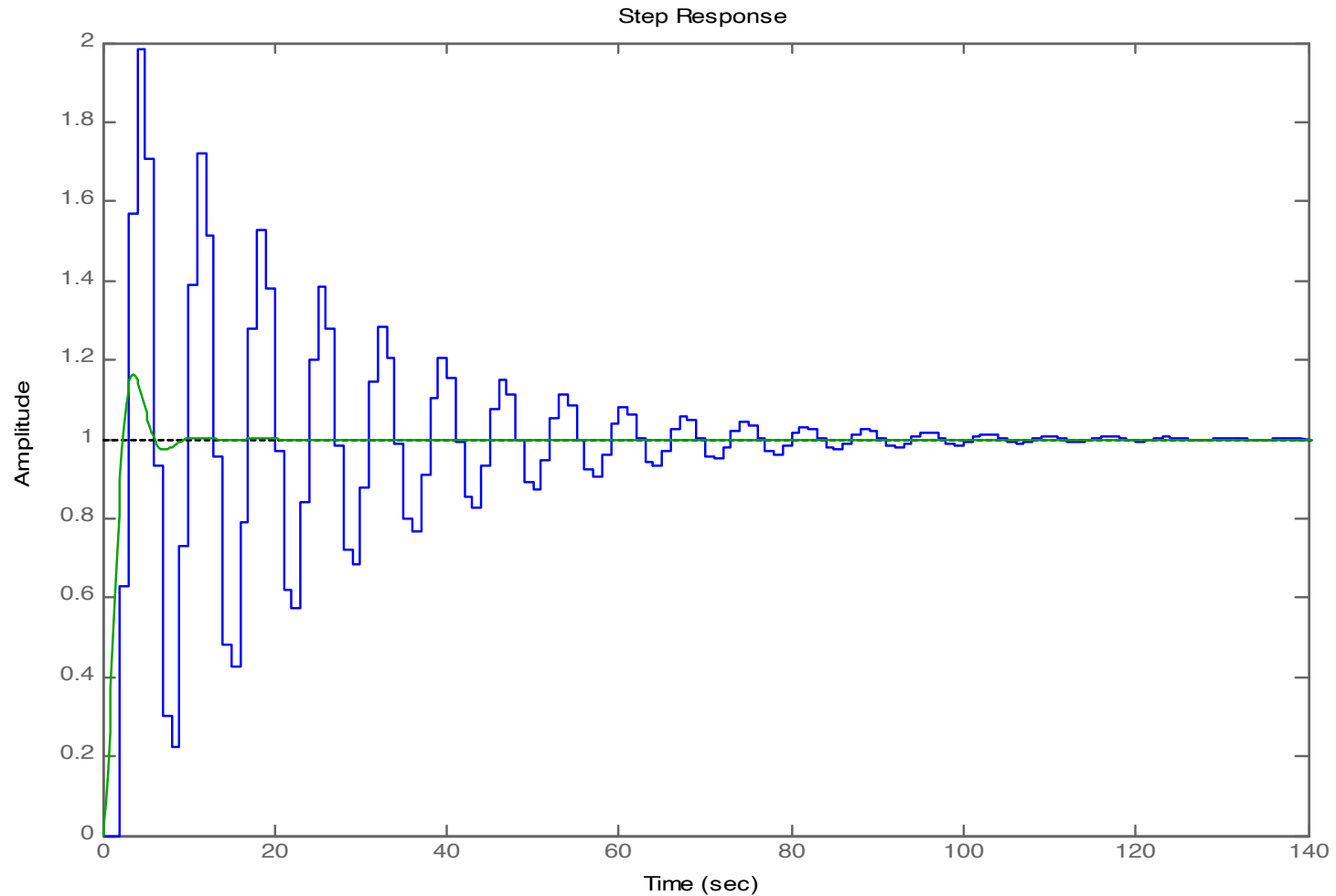
Oscilações Escondidas

$$D(z) = 9 \frac{(z - 0.8)}{(z + 0.8)}$$



Latência

$$D(z) = 13 \frac{(z - 0.88)}{z(z + 0.5)} \quad \text{Função Estritamente Própria}$$



Projeto com pólo na origem $D(z) = 6 \frac{(z - 0.83)}{z(z + 0.9)}$

